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Twin Supergravities from Yang-Mills Squared

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We consider 'twin supergravities' - pairs of supergravities with $\mathcal{N}_+$ and $\mathcal{N}_-$ supersymmetries, $\mathcal{N}_+ > \mathcal{N}_-$, with identical bosonic sectors - in the context of tensoring super Yang-Mills multiplets. It is demonstrated that the pairs of twin supergravity theories are related through their left and right super Yang-Mills factors. This procedure generates new theories from old. In particular, the matter coupled $\mathcal{N}_-$ twins in $D = 3, 5, 6$ and the $\mathcal{N}_- = 1$ twins in $D = 4$ have not, as far as we are aware, been obtained previously using the double-copy construction, adding to the growing list of double-copy constructible theories. The use of fundamental matter multiplets in the double-copy construction leads us to introduce a bi-fundamental scalar that couples to the well-known bi-adjoint scalar field. It is also shown that certain matter coupled supergravities admit more than one factorisation into left and right super Yang-Mills-matter theories.

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I. INTRODUCTION

It has been known for some time [1, 2] that there exist pairs of supergravity theories with identical bosonic sectors, both in terms of content and couplings, but distinct degrees of supersymmetry $\mathcal{N}_+ > \mathcal{N}_-$. The canonical example in $D = 4$ dimensions is given by $\mathcal{N}_+ = 6$ supergravity and the magic $\mathcal{N}_- = 2$ supergravity coupled to 15 vector multiplets, which despite their distinct supersymmetric completions have the same bosonic Lagrangian with scalar coset $\mathrm{SO}^\star(12)/\,\mathrm{U}(6)$. All such 'twin' supergravities, which we will denote by $(\mathcal{N}_+, \mathcal{N}_-)$, were classified in [3, 4].

A so far unrelated idea is that of gravity as 'the square of gauge theory'. Schematically[1],

$$A_\mu \otimes \tilde{A}_\nu = g_{\mu\nu} \oplus B_{\mu\nu} \oplus \varphi. \tag{1}$$

Here, A_μ and $\tilde{A}_\nu$ are the gauge potentials of two distinct Yang-Mills theories, which we will refer to as left (no tilde) and right (tilde), respectively. They can have arbitrary and independent non-Abelian gauge groups G and $\tilde{G}$. Beyond the identification of asymptotic on-mass-shell states, this formal identity can be motivated by the Kawai-Lewellen-Tye (KLT) relations, which connect tree-level amplitudes of closed strings to sums of products of open string amplitudes [7]. More recently, invoking Bern-Carrasco-Johansson (BCJ) colour-kinematic duality [8] it has been conjectured [9, 10] that the on-mass-shell momentum-space scattering amplitudes for gravity are the "double-copy" of gluon scattering amplitudes in Yang-Mills theory to all orders in perturbation theory. These relations have since been generalised to a large class of (super)gravity theories, including a variety of matter couplings and even pure gravity [11–19]. For reviews see [20, 21]. The double-copy prescription has proven itself a tremendously effective computational tool, pushing the boundaries of what can be achieved in perturbative quantum gravity and in the process revealing numerous surprises [22–30]. There is now a growing literature [5, 6, 31–76] expanding upon, and refining our understanding of, these remarkable relations, which has benefitted from a diverse array of complementary approaches: scattering equations [34–42], kinematic algebras [43–47], string theory [48–58], twistor theory [59–64], on-shell and off-shell symmetries [5, 6, 65–70], minimal physical assumptions [71] and even non-perturbative classical solutions [72–76].

In the present paper we look at twin supergravities in this context, studying the relationship between the two Yang-Mills factors generating the twin supergravity theories. In [66, 68, 70] all possible products of two super Yang-Mills theories, with *no* additional matter couplings, in dimensions $3 \leq D \leq 10$ were considered yielding a pyramid of supergravity theories and their corresponding scalar manifolds. See Table IV. Remarkably, all theories appearing in the pyramid, with the exception of the maximal "spine" indicated in red, have a twin with fewer supersymmetries.

It is demonstrated here that such twin supergravity theories are related in a controlled manner through their Yang-Mills factors. Each twin pair $(\mathcal{N}_+, \mathcal{N}_-)$ can be regarded as a pair of complementary consistent truncations of a single $(\mathscr{N} = \mathcal{N}_+ + \mathcal{N}_-)$-extended parent supergravity theory [3], which is given by the product of a left $\mathcal{N}$-extended Yang-Mills theory with a right $\tilde{\mathcal{N}}$-extended Yang-Mills theory, where $\mathscr{N} = \mathcal{N} + \tilde{\mathcal{N}}$ and without loss of generality we consider $\mathcal{N} \geq \tilde{\mathcal{N}}$. The twins relations are mediated by the Yang-Mills factors as depicted schematically here:

$$\begin{tikzcd}
 & \begin{array}{c}\text{Parent supergravity}\\ \mathbf{G}_{\mathcal{N}+\tilde{\mathcal{N}}} \oplus \mathbf{M}_{\mathcal{N}+\tilde{\mathcal{N}}}\end{array} \arrow[d, "\text{Yang-Mills factors}"] & \\
 & \mathbf{V}_{\mathcal{N}} \otimes \tilde{\mathbf{V}}_{\tilde{\mathcal{N}}} \arrow[dl] \arrow[dr] & \\
 {[\mathbf{V}_{\mathcal{N}'} \oplus \mathbf{C}^{\rho}_{\mathcal{N}'}] \otimes \tilde{\mathbf{V}}_{\tilde{\mathcal{N}}}} \arrow[rr, dashed, leftrightarrow, "\text{twin relation}"'] \arrow[d] & & {[\mathbf{V}_{\mathcal{N}'} \oplus \mathbf{C}^{\rho}_{\mathcal{N}'}] \otimes [\tilde{\mathbf{V}}_{\tilde{\mathcal{N}}'} \oplus \tilde{\mathbf{C}}^{\tilde{\rho}}_{\tilde{\mathcal{N}}'}]} \arrow[d] \\
 \begin{array}{c}\mathcal{N}_+ \text{ big twin supergravity}\\ \mathbf{G}_{\mathcal{N}_+} \oplus \mathbf{M}_{\mathcal{N}_+}\end{array} & & \begin{array}{c}\mathcal{N}_- \text{ little twin supergravity}\\ \mathbf{G}_{\mathcal{N}_-} \oplus \mathbf{M}_{\mathcal{N}_-}\end{array}
\end{tikzcd} \tag{2}$$

Here, $\mathbf{G}_{\mathcal{N}}, \mathbf{V}_{\mathcal{N}}, \mathbf{C}_{\mathcal{N}}$ and $\mathbf{M}_{\mathcal{N}}$ denote $\mathcal{N}$-extended gravity, vector, spinor and generic (not necessarily irreducible) matter multiplets, respectively. The left $\mathbf{V}_{\mathcal{N}}$ and right $\tilde{\mathbf{V}}_{\tilde{\mathcal{N}}}$ multiplets of the parent supergravity are decomposed into $\mathcal{N}' < \mathcal{N}$ and $\tilde{\mathcal{N}}' < \tilde{\mathcal{N}}$ multiplets, $\mathbf{V}_{\mathcal{N}'} \oplus \mathbf{C}^{\rho}_{\mathcal{N}'} \oplus \cdots$ and $\tilde{\mathbf{V}}_{\tilde{\mathcal{N}}'} \oplus \tilde{\mathbf{C}}^{\tilde{\rho}}_{\tilde{\mathcal{N}}'} \oplus \cdots$, and the resulting adjoint spinor

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[1] The precise meaning of this product and the issue of the gauge indices is discussed in [5, 6] and given in (23).

multiplets are replaced by fundamental[2] multiplets as indicated by the superscript ρ. This procedure generates new theories from old. In particular, the matter coupled $\mathcal{N}_-$ twins in $D = 3, 5, 6$ and the $\mathcal{N}_- = 1$ twins in $D = 4$ have not, as far as we are aware, been obtained previously using the double-copy construction, adding to the growing list of double-copy constructible theories. The twin theories in $D = 3, 4, 5, 6$ and their left/right (super) Yang-Mills factorisations as determined by the above prescription are given in Table V, Table VI, Table VII and Table VIII. The use of fundamental matter multiplets in the double-copy construction leads us to introduce a bi-fundamental scalar that couples to the well-known bi-adjoint scalar field. It is also shown that certain matter coupled supergravities admit more than one factorisation into left and right super Yang-Mills-matter theories.

The remaining sections are organised as follows. In section II we review the classification of all twin supergravities. In section III we demonstrate how the pyramid twins are related via Yang-Mills squared. We outline the general procedure and discuss the bi-fundamental scalar theory before presenting a detailed example in section III A. In section III B we summarise the pyramid of twins and make some additional comments on the $D = 6$ and $D = 3$ cases. The *triplets* are considered section III C. In section III D we treat the isolated twin pair not appearing in the pyramid and the generalisation of the $D = 4, (2, 1)$ twin pair to a sequence of $D = 4, (2, 1)$ twin pairs. We conclude in section IV with a summary and future directions.

II. TWIN SUPERGRAVITY THEORIES

Twin supergravities are theories that have identical bosonic Lagrangians, but different supersymmetric completions. Twin pairs are denoted by $(\mathcal{N}_+, \mathcal{N}_-)$ with $\mathcal{N}_+ > \mathcal{N}_-$. Such theories appear in $D = 3, 4, 5, 6$. In $D = 3$ where all vectors dualise to scalars, matching of the scalar cosets is a sufficient condition for twinness. However, this criterion is necessary but not sufficient in $D = 4, 5, 6$.

Here we summarise the classification of twin supergravities provided in [3, 4]. The classification is done in $D = 3$ and relies on listing the scalar manifolds of theories with different $\mathcal{N}$ and then checking whether any of them match. The scalar manifolds of $D = 3$ supergravities are given in Table I where we distinguish between the theories with Kähler and Quaternionic manifolds, the matter-coupled theories and the unique pure supergravity theories. All scalar

$\mathcal{N}$	$\mathcal{M}_{scalar(3)}$
1	Riemannian
2	Kähler
3	Quaternionic
4	Quaternionic $\times$ Quaternionic
5	$\frac{\mathrm{USp}(1,n)}{\mathrm{U}(1)\times\mathrm{USp}(n)}$
6	$\frac{\mathrm{SU}(4,n)}{\mathrm{SU}(4)\times\mathrm{SU}(n)\times\mathrm{U}(1)}$
8	$\frac{\mathrm{SO}(8,n)}{\mathrm{SO}(8)\times\mathrm{SO}(n)}$
9	$\frac{F_{4(-20)}}{\mathrm{SO}(9)}$
10	$\frac{E_{6(-14)}}{\mathrm{SO}(10)\times\mathrm{SO}(2)}$
12	$\frac{E_{7(-5)}}{\mathrm{SO}(12)\times\mathrm{SO}(3)}$
16	$\frac{E_{8(8)}}{\mathrm{SO}(16)}$

TABLE I. The scalar manifolds of $D = 3$ supergravity theories with $\mathcal{N}$ supercharges.

manifolds in Table I are Riemannian and therefore all theories could be thought as $\mathcal{N} = 1$ theories. Such twins are

[2] We use fundamental here loosely to refer to gauge group representations other than the adjoint.

in this sense trivial. There are, however, a number that are particularly natural from the perspective of Yang-Mills squared, which we will therefore include in our analysis. Furthermore, all $\mathcal{N}=3$ theories can be interpreted as $\mathcal{N}=4$ with a trivial second quaternionic factor and thus are omitted in the analysis that follows. In $D=3$, where all vectors are dual to scalars, two theories that have the same scalar manifold are twins. To carry out the classification one needs to check whether any of the scalar manifolds for $\mathcal{N}\geq 5$ are Kähler, quaternionic or both. A list of the possible Kähler and Quaternionic manifolds is provided in appendix A. A matching of the scalar manifolds gives the classification provided in [3].

$(\mathcal{N}_+,\mathcal{N}_-)$	$\mathcal{M}_{scalar}$	D_{max}
$(4,2)$	$\frac{\mathrm{SU}(2,p)}{\mathrm{SU}(2)\times\mathrm{U}(p)}\times\frac{\mathrm{SU}(2,q)}{\mathrm{SU}(2)\times\mathrm{U}(q)}$	4
$(6,2)$	$\frac{\mathrm{SU}(4,p)}{\mathrm{SU}(4)\times\mathrm{U}(p)}$	4^*
$(8,2)$	$\frac{\mathrm{SO}(8,2)}{\mathrm{SO}(8)\times\mathrm{SO}(2)}$	4
$(10,2)$	$\frac{E_{6(-14)}}{\mathrm{SO}(10)\times\mathrm{SO}(2)}$	4
$(5,4)$	$\frac{\mathrm{USp}(2,1)}{\mathrm{USp}(2)\times\mathrm{U}(1)}$	3
$(8,4)$	$\frac{\mathrm{SO}(8,4)}{\mathrm{SO}(8)\times\mathrm{SO}(4)}$	6
$(12,4)$	$\frac{E_{7(-5)}}{\mathrm{SO}(12)\times\mathrm{SO}(3)}$	6

TABLE II. The twin supergravity theories in $D=3$. D_{max} is the highest dimension to which these theories can be uplifted. * The (6,2) sequence admits an uplift to D=4 only for $p=2$, since for all other values of p it oxidises to $\mathcal{N}=3$, $D=4$ theories whose kinetic vector matrix in non-holomorphic, which cannot be twins to $\mathcal{N}=1$. This refines the treatment in [3].

The classification provided in Table II differs from the one in [3] in the $(4,2)$ entry where the authors give only one of the two factors. Since the product of two Kähler manifolds is Kähler the second factor is allowed. As the authors mention, it is clear from Table II that the theory with scalar manifold $\mathrm{SU}(4,2)/\,\mathrm{SU}(4)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ has three supersymmetric completions and we refer to it as the triplet $(6,4,2)$. More generally, triplets will be denoted $(\mathcal{N}_+,\mathcal{N}_-^+,\mathcal{N}_-^-)$, where $\mathcal{N}_+$ is the big sibling of both $\mathcal{N}_-^\pm$. The two cases, $(\mathcal{N}_+,\mathcal{N}_-^+)$ and $(\mathcal{N}_+,\mathcal{N}_-^-)$, are conventional twin pairs in the sense that they follow the same pattern as the pure twins, as described in section III C.

All twin pairs in higher dimensions can be obtained by oxidation of the $D=3$ ones [3]. We now oxidise the theories from $D=3\to 4$ (halves $\mathcal{N}$), $D=4\to 5$ (preserves $\mathcal{N}$) and $D=5\to 6$ (halves $\mathcal{N}$) to obtain the twin pairs together with their scalar manifolds in the higher dimensional theories. We do so to demonstrate the crucial point that *although all twin pairs can be obtained from oxidation, not all oxidised pairs form twins.* Matching bosonic content and scalar manifolds is a necessary but not sufficient condition in $D>3$. A simple example illustrating this point in $D=4$ is given by the scalar manifold $\mathrm{SU}(3,3)/[\mathrm{U}(3)\times\mathrm{SU}(3)]$, which occurs three times: (*i*) in $\mathcal{N}=1$, (*ii*) in $\mathcal{N}=3$ and (*iii*) in $\mathcal{N}=2$. These theories have 64, 64, 80 degrees of freedom respectively and thus clearly the latter cannot form a twin pair with either of the other two. The $\mathcal{N}=3$ theory is coupled to three vector multiplets and the kinetic vector multiplet is non-holomorphic. Since all $\mathcal{N}=1$ supergravities in $D=4$ must have a holomorphic kinetic vector matrix, the first two theories cannot form a twin pair either.

A second example in $D=4$ that serves to highlight the various subtleties is given by the coset $\mathrm{SU}(1,1)/\,\mathrm{U}(1)$. For $\mathcal{N}\geqslant 2$ the scalar manifold $\mathrm{SU}(1,1)/\,\mathrm{U}(1)$ occurs three times:

1. $\mathcal{N}=4$ pure supergravity [80]: the U-duality group is $\mathrm{U}(4)\simeq\mathrm{SO}(6)\times\mathrm{SU}(1,1)$, with scalar manifold,

$$\mathrm{SU}(1,1)/\,\mathrm{U}(1)\times\mathrm{SO}(6)/\,\mathrm{SO}(6)\cong\mathrm{SU}(1,1)/\,\mathrm{U}(1). \tag{3}$$

As such, this is the $n=0$ element of the sequence $\mathrm{SU}(1,1)/\,\mathrm{U}(1)\times\mathrm{SO}(6,n)/\,[\mathrm{SO}(6)\times\mathrm{SO}(n)]$, where n denotes the number of vector multiplets coupled to the $\mathcal{N}=4$ gravity multiplet. The Abelian 2-form field strengths and their duals sit in the $(\mathbf{6},\mathbf{2})$ of $\mathrm{SO}(6)\times\mathrm{SU}(1,1)$. The pair $(\mathrm{SO}(6)\times\mathrm{SU}(1,1),(\mathbf{6},\mathbf{2}))$ defines a "group of type

E_7" [81] of a very particular character [82]: while in the bare charges basis, the invariant is quartic, it becomes a perfect square in the dressed (supersymmetry) charges basis[3].

2. $\mathcal{N}=2$ supergravity *minimally coupled* to a single vector multiplet (*i.e.* the $\mathcal{N}=2$ axion-dilaton model) [83]: the U-duality group is $\mathrm{U}(1,1)$ with scalar manifold,

$$\overline{\mathbb{CP}}^1 \cong \mathrm{SU}(1,1)/\,\mathrm{U}(1)\times \mathrm{U}(1)/\,\mathrm{U}(1) \cong \mathrm{SU}(1,1)/\,\mathrm{U}(1). \tag{4}$$

As such, this is the $n=1$ element of the sequence $\overline{\mathbb{CP}}^n \cong \mathrm{U}(1,n)/\left[\mathrm{U}(1)\times \mathrm{U}(n)\right]$, where n denotes the number of vector multiplets minimally coupled to the $\mathcal{N}=2$ gravity multiplet. The Abelian 2-form field strengths and their duals sit in the $\mathbf{2}_1+\mathbf{2}_{-1}$ of $\mathrm{U}(1,1)$. The global $\mathrm{U}(1)$ factor is a relic of the compact symmetry of the Maxwell theory of the lone graviphoton to which the electromagnetic sector reduces if the vector multiplet is truncated. Note that the axion-dilaton model is a consistent truncation of the $\mathcal{N}=4$ pure supergravity considered at point 1. In the bosonic sector this amounts to removing four graviphotons out of six; in this way, the $\mathcal{N}=2$ axion-dilaton model is obtained in a symplectic frame in which the holomorphic prepotential reads $F=-iX^0X^1$, as opposed to the manifestly $\mathrm{SU}(1,1)$-symmetric Fubini-Study symplectic frame for which,

$$F=-\frac{i}{2}\left[\left(X^0\right)^2-\left(X^1\right)^2\right]. \tag{5}$$

The two frames are related by a global $\mathrm{USp}\left(4,\mathbb{R}\right)$ transformation [84]. The pair $(\mathrm{U}(1,1),\mathbf{2}_1+\mathbf{2}_{-1})$ defines a *degenerate* group of type E_7 [85].

3. $\mathcal{N}=2$ supergravity coupled to a single vector multiplet via a cubic pre-potential (often referred to as the T^3 model and most simply obtained by dimensionally reducing minimal $D=5$ supergravity): the U-duality group is $\mathrm{SU}(1,1)$, with no additional global compact factors present. The Abelian 2-form field strengths and their duals sit in the $\mathbf{4}$ of $\mathrm{SU}(1,1)$. The manifold $\mathrm{SU}(1,1)/\,\mathrm{U}(1)$ is an isolated case in the classification of symmetric special Kähler spaces [86, 87]. Indeed, under dimensional reduction to $D=3$, this is mapped to the exceptional quaternionic manifold $G_{2(2)}/\,\mathrm{SO}(4)$. The pair $(\mathrm{SU}(1,1),\mathbf{4})$ provides the simplest example of a group of type E_7.

Although all share the same scalar coset none are twin. Firstly, the $\mathcal{N}=4$ theory has 32 degrees of freedom, while the two $\mathcal{N}=2$ theories each have 16. Secondly, despite having the same bosonic (and fermionic) content and scalar manifolds the two $\mathcal{N}=2$ theories have distinct couplings in the bosonic sector since the Abelian field strengths and their duals transform in two distinct representations of $\mathrm{SU}(1,1)$, the $\mathbf{2}+\mathbf{2}$ and $\mathbf{4}$, respectively.

Similarly, for $\mathcal{N}=1$ supergravity the scalar coset $\mathrm{SU}(1,1)/\,\mathrm{U}(1)$ considered above also appears in at least three examples. These belong to five families of $\mathcal{N}=1$ supergravity theories with scalar manifolds compatible with non-trivial electromagnetic duality as given in [77]:

1. $\mathrm{USp}(2n,\mathbb{R})/\,\mathrm{U}(n)$ coupled to n vector multiplets, with duality group $\mathrm{Sp}(n,\mathbb{R})$.
2. $\mathrm{U}(1,n)/\,\mathrm{U}(n)$ coupled to $n+1$ vector multiplets, with duality group $\mathrm{U}(1,n)\subset \mathrm{Sp}(n+1,\mathbb{R})$.
3. $\mathrm{SU}(1,1)/\,\mathrm{U}(1)$ coupled to n vector multiplets, with duality group $\mathrm{SL}(2,\mathbb{R})\times \mathrm{SO}(n)\subset \mathrm{Sp}(n,\mathbb{R})$
4. $\mathrm{SO}(2,n)/\,\mathrm{SO}(2)\times \mathrm{SO}(n)$ coupled to r vector multiplets in the r-dimensional spinor representation of $\mathrm{SO}(1,n-1)\subset \mathrm{SO}(2,n)$, with duality group $\mathrm{Spin}(2,n)\subset \mathrm{Sp}(r,\mathbb{R})$
5. $\mathrm{U}(n,n)/\,\mathrm{U}(n)\times \mathrm{U}(n)$ coupled to $2n$ vector multiplets.

The scalar coset $\mathrm{SU}(1,1)/\,\mathrm{U}(1)$ occurs for: (*i*) classes 1 and 4 coupled to a single vector multiplet with U-duality group $\mathrm{Sp}(1,\mathbb{R})\cong \mathrm{SU}(1,1)$, (*ii*) classes 2 and 5 coupled to two vector multiplets with U-duality group $\mathrm{U}(1,1)\subset \mathrm{Sp}(4,\mathbb{R})$ and (*iii*) class 3 coupled to n vector multiplets with U-duality group $\mathrm{SL}(2,\mathbb{R})\times \mathrm{SO}(n)$. Note, case (*ii*) has an additional $\mathrm{U}(1)$ under which the scalars are neutral. The complex scalar of case (*iii*) does not exhibit an attractor behaviour for spherically symmetric stationary black hole solutions [77].

These $\mathcal{N}=1$ theories with $\mathrm{SU}(1,1)/\,\mathrm{U}(1)$ coset do constitute twins of pure $\mathcal{N}=4$ supergravity and the $\mathcal{N}=2$ axion-dilaton model, but not the T^3 model as its kinetic vector matrix is not holomorphic, see for example (2.8) and (2.9) of [88]. In particular, case (*iii*) with $n=6$ is twinned with pure $\mathcal{N}=4$ supergravity. Dimensional reduction to $D=3$, accompanied by the dualisation of all vectors, maps the scalar $D=4$ coset $\mathrm{SU}(1,1)/\,\mathrm{U}(1)$ to

[3] The only other known example of a group of type E_7 sharing this property is the pair $(\mathrm{SU}(5,1),\mathbf{20})$ of $\mathcal{N}=5$, $D=4$ supergravity [82].

$(\mathcal{N}_+,\mathcal{N}_-)_{(4)}$	$\mathcal{M}_{scalar(4)}$	$(\mathcal{N}_+,\mathcal{N}_-)_{(5)}$	$\mathcal{M}_{scalar(5)}$	$(\mathcal{N}_+,\mathcal{N}_-)_{(6)}$	$\mathcal{M}_{scalar(6)}$
$(2,1)$	$\frac{\mathrm{U}(1,p-1)}{\mathrm{U}(1)\times\mathrm{U}(p-1)}\times\frac{\mathrm{SU}(2,q)}{\mathrm{SU}(2)\times\mathrm{U}(q)}$				
$(3,1)$	$\frac{\mathrm{U}(3,1)}{\mathrm{U}(1)\times\mathrm{U}(3)}$				
$(4,1)$	$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$				
$(5,1)$	$\frac{\mathrm{SU}(5,1)}{\mathrm{U}(5)}$				
$(4,2)$	$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\times\frac{\mathrm{SO}(6,2)}{\mathrm{U}(4)}$	$(4,2)$	$\mathrm{SO}(1,1)\times\frac{\mathrm{SO}(5,1)}{\mathrm{USp}(2)}$	$((1,1),(0,1))$	$\frac{\mathrm{O}(1,1)\times\mathrm{Sp}(1)^2}{\mathrm{U}(1)^2}$
				$((2,0),(0,1))$	$\frac{\mathrm{SU}^\star(4)}{\mathrm{USp}(2)}$
$(6,2)$	$\frac{\mathrm{SO}^\star(12)}{\mathrm{U}(6)}$	$(6,2)$	$\frac{\mathrm{SU}^\star(6)}{\mathrm{USp}(3)}$	$((2,1),(0,1))$	$\frac{\mathrm{SU}^\star(4)\times\mathrm{Sp}(1)}{\mathrm{USp}(2)\times\mathrm{U}(1)}$

TABLE III. The twin supergravities theories in $D = 4, 5, 6$.

$\mathrm{SO}(8,2)/[\mathrm{SO}(8)\times\mathrm{SO}(2)]$. Similarly, case (*ii*) is twinned with the $\mathcal{N}=2$ axion-dilaton model, but in this case the coset $\mathrm{SU}(1,1)/\mathrm{U}(1)$ is mapped to the quaternionic Kähler manifold $\mathrm{SU}(2,2)/[\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)]$ in $D=3$.

The classification of twin supergravity theories is provided in Table III. First we oxidise the twin pairs to $D=4$. At this point one would expect to get an infinite sequence of $(3,1)$ pairs with scalar manifold $\mathrm{U}(3,p-1)/\mathrm{U}(1)\times\mathrm{U}(3)\times\mathrm{U}(p-1)$. The $\mathcal{N}=3$ theory is coupled to $p-1$ vector multiplets. The kinetic vector matrix of matter-coupled $\mathcal{N}=3$, $D=4$ supergravity has been computed in Appendix C of [89], as well as more recently in [82] (*cfr.* Section 4 and in particular (4.10) and (4.14) therein). As following from [82, 89] this matrix is always non-holomorphic, except in the case of one vector multiplet[4]. Thus the only $(3,1)$ pair is the one belonging to the triplet $(3,2,1)$ with scalar coset $\mathrm{U}(3,1)/\mathrm{U}(3)\times\mathrm{U}(1)\times\mathrm{U}(1)$, again illustrating the point that for $D>3$ matching bosonic content and scalar cosets is not sufficient. Oxidising the twin pairs to $D=5$ is straightforward. Each twin pair oxidises to a unique pair of $D=5$ twin supergravities. Finally, we oxidise to $D=6$. One needs to be careful as both theories in the $(4,2)$ pair admit two different oxidations. The $\mathcal{N}_-=2$ theory can be interpreted as part of the sequence $L(P,0)$ (with $P=4$)[5] or as part of the sequence $L(q,0)$ (with $q=4$)[6] [87, 90, 91]. In the former case it uplifts to chiral $(0,1)$ theory coupled to 1 tensors and 4 vector multiplets, with U-duality group $\mathrm{O}(1,1)\times\mathrm{SO}(4)$. In the latter case it uplifts again to $(0,1)$ but now coupled to 5 tensor multiplets and U-duality group $\mathrm{SU}^\star(4)$. The $\mathcal{N}_+=4$ partner can uplift to either a $(2,0)$ or a $(1,1)$ theory [92]. The former is coupled to 1 tensor multiplet while the latter is “pure" supergravity. The respective U-duality groups are $\mathrm{O}(1,1)\times\mathrm{SO}(4)$ and $\mathrm{SU}^\star(4)$. This explains the two slots appearing in the last column of Table III.

III. TWINS FROM SUPER YANG-MILLS SQUARED

In this section we describe the Yang-Mills squared origin of the twin supergravities. We begin by considering the pyramid of supergravity theories given in Table IV. It is generated by the product of a left and right super Yang-Mills multiplet in $3\le D\le 10$,

$$\mathbf{V}_{\mathcal{N}}\otimes\tilde{\mathbf{V}}_{\tilde{\mathcal{N}}}=\mathbf{G}_{\mathcal{N}+\tilde{\mathcal{N}}}\oplus\mathbf{M}_{\mathcal{N}+\tilde{\mathcal{N}}},\tag{6}$$

where $\mathbf{V},\tilde{\mathbf{V}}$ are in the adjoint representation of G and $\tilde{G}$. See [66, 68, 70] for details. When a point in the pyramid $\mathbf{G}_{\mathcal{N}+\tilde{\mathcal{N}}}\oplus\mathbf{M}_{\mathcal{N}+\tilde{\mathcal{N}}}$ admits a twin we will denote the pair by

$$\mathbf{G}_{\mathcal{N}_+}\oplus\mathbf{M}_{\mathcal{N}_+}\quad\underset{\text{twins}}{\longleftrightarrow}\quad\mathbf{G}_{\mathcal{N}_-}\oplus\mathbf{M}_{\mathcal{N}_-},\tag{7}$$

[4] We would like to thank Alessandra Gnecchi for useful correspondence concerning this point.
[5] This theory is anomaly-free only when coupled to 248 hyper multiplets.
[6] This theory is anomaly-free only when coupled to 128 hyper multiplets.

or simply $(\mathcal{N}_+, \mathcal{N}_-)$. In Table IV we have summarised the twin pairs $(\mathcal{N}_+, \mathcal{N}_-)$ appearing in the pyramid. Note, since there are no twins for $D > 6$ we have truncated the pyramid at $D = 6$. We see that all theories obtained via (6), excluding the maximal supergravities living on the spine for $D > 3$, have a twin.

Not only do all the non-maximal supergravity theories obtained as the square of pure super Yang-Mills admit a twin, the two theories are related in a controlled manner through their double-copy constructions. First note that $(\mathcal{N}_+, \mathcal{N}_-)$ can be regarded as a pair of complementary consistent truncations of a single $\mathscr{N}$-extended parent supergravity theory given by,

$$\mathbf{V}_{\mathcal{N}} \otimes \tilde{\mathbf{V}}_{\tilde{\mathcal{N}}} = \mathbf{G}_{\mathcal{N}+\tilde{\mathcal{N}}} \oplus \mathbf{M}_{\mathcal{N}+\tilde{\mathcal{N}}}, \qquad \text{where} \qquad \mathscr{N} = \mathcal{N} + \tilde{\mathcal{N}} = \mathcal{N}_+ + \mathcal{N}_-. \tag{8}$$

This follows from simple symmetry requirements. The $\mathcal{N}_+$ R-symmetry is necessarily a subgroup of the parent $\mathscr{N}$ R-symmetry[7]. On the other hand, for the $\mathcal{N}_-$ twin the same group is repurposed as matter isotropy group and we have to further include the $\mathcal{N}_-$ R-symmetry in the parent R-symmetry implying that the minimal $\mathscr{N}$ is given by $\mathcal{N}_+ + \mathcal{N}_-$.

Let us now summarise the twin double-copy procedure for $(\mathcal{N}_+, \mathcal{N}_-)$:

1. Decompose $\mathbf{V}_{\mathcal{N}} = \mathbf{V}_{\mathcal{N}'} \oplus \mathbf{C}_{\mathcal{N}'}$ where $\mathcal{N}_+ = \mathcal{N}' + \tilde{\mathcal{N}}$.

2. Replace the adjoint spinor multiplet by a fundamental spinor multiplet, $\mathbf{C}_{\mathcal{N}'} \to \mathbf{C}^{\rho}_{\mathcal{N}'}$, carrying a representation ρ of the left gauge group and thus reducing the degree of supersymmetry to $\mathcal{N}' < \mathcal{N}$.

3. The double-copy construction
$$[\mathbf{V}_{\mathcal{N}'} \oplus \mathbf{C}^{\rho}_{\mathcal{N}'}] \otimes \tilde{\mathbf{V}}_{\tilde{\mathcal{N}}} = \mathbf{V}_{\mathcal{N}'} \otimes \tilde{\mathbf{V}}_{\tilde{\mathcal{N}}}, \tag{9}$$
yields the $\mathcal{N}_+ = \mathcal{N}' + \tilde{\mathcal{N}}$ twin as a truncation of the parent supergravity through its Yang-Mills factors by discarding the states that would have arisen from $\mathbf{C}_{\mathcal{N}'} \otimes \tilde{\mathbf{V}}_{\tilde{\mathcal{N}}}$, in particular $\mathcal{N} - \mathcal{N}'$ of the gravitini.

4. Decompose
$$\tilde{\mathbf{V}}_{\tilde{\mathcal{N}}} = \tilde{\mathbf{V}}_{\tilde{\mathcal{N}}'} \oplus \tilde{\mathbf{C}}_{\tilde{\mathcal{N}}'} \oplus \cdots \tag{10}$$
where $\mathcal{N}_- = \mathcal{N}' + \tilde{\mathcal{N}}'$. Note that typically $\tilde{\mathcal{N}}' = 0$ so that $\tilde{\mathbf{V}}_{\tilde{\mathcal{N}}}$ decomposes as
$$\tilde{\mathbf{V}}_{\tilde{\mathcal{N}}} = \tilde{A} \oplus \tilde{\chi} \oplus \tilde{\phi}, \tag{11}$$
where we regard χ as an $\mathcal{N} = 0$ spinor multiplet $\mathbf{C}_0$.

5. Replace all adjoint spinor multiplets by fundamental spinor multiplets, $\tilde{\mathbf{C}}_{\tilde{\mathcal{N}}'} \to \tilde{\mathbf{C}}^{\tilde{\rho}}_{\tilde{\mathcal{N}}'}$, carrying a representation $\tilde{\rho}$ of the right gauge group $\tilde{G}$ and thus reducing the degree of supersymmetry to $\tilde{\mathcal{N}}' < \tilde{\mathcal{N}}$.

6. The double-copy construction
$$[\mathbf{V}_{\mathcal{N}'} \oplus \mathbf{C}^{\rho}_{\mathcal{N}'}] \otimes [\tilde{\mathbf{V}}_{\tilde{\mathcal{N}}'} \oplus \tilde{\mathbf{C}}^{\tilde{\rho}}_{\tilde{\mathcal{N}}'} \oplus \cdots] = [\mathbf{V}_{\mathcal{N}'} \otimes \tilde{\mathbf{V}}_{\tilde{\mathcal{N}}'}] \oplus [\mathbf{C}^{\rho}_{\mathcal{N}'} \otimes \tilde{\mathbf{C}}^{\tilde{\rho}}_{\tilde{\mathcal{N}}'}] \oplus [\mathbf{V}_{\mathcal{N}'} \otimes \cdots], \tag{12}$$
yields the $\mathcal{N}_- = \mathcal{N}' + \tilde{\mathcal{N}}'$ twin as a truncation of the parent supergravity through its Yang-Mills factors. Note, by using $\tilde{\mathbf{C}}^{\tilde{\rho}}_{\tilde{\mathcal{N}}'}$ we discard $\tilde{\mathcal{N}} - \tilde{\mathcal{N}}'$ out of the $\mathcal{N}' + \tilde{\mathcal{N}}$ gravitini of the $\mathcal{N}_+$ twin that would have arisen from $\mathbf{V}_{\mathcal{N}'} \otimes \tilde{\mathbf{C}}_{\tilde{\mathcal{N}}}$, as well as a subset of the spinor states. They are replaced by the spinors arising from $\mathbf{C}^{\rho}_{\mathcal{N}'} \otimes \tilde{\mathbf{C}}^{\tilde{\rho}}_{\tilde{\mathcal{N}}'}$.

In summary the two twin theories with $\mathcal{N}_+ = \mathcal{N}' + \tilde{\mathcal{N}}$ and $\mathcal{N}_- = \mathcal{N}' + \tilde{\mathcal{N}}'$ are related through,

$$[\mathbf{V}_{\mathcal{N}'} \oplus \mathbf{C}^{\rho}_{\mathcal{N}'}] \otimes \tilde{\mathbf{V}}_{\tilde{\mathcal{N}}} \longleftrightarrow [\mathbf{V}_{\mathcal{N}'} \oplus \mathbf{C}^{\rho}_{\mathcal{N}'}] \otimes [\tilde{\mathbf{V}}_{\tilde{\mathcal{N}}'} \oplus \mathbf{C}^{\tilde{\rho}}_{\tilde{\mathcal{N}}'} \oplus \cdots]. \tag{13}$$

The resulting twin theories in $D = 3, 4, 5, 6$ are given in Table V, Table VI, Table VII and Table VIII. The $D = 4, (6, 2)$ example is given in full detail in section III A, where some of the subtleties of the above sketch are addressed. In particular:

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[7] When $\mathscr{N} \leq 4$ there can be an additional isotropy group, which complicates the argument but does not change the conclusion.

a) Multiplets carrying the adjoint (fundamental) representation of the left gauge group only double-copy with multiplets carrying the adjoint (fundamental) representation of the right gauge group, leading to a *sum of squares*. For example:

$$[\mathbf{V}_{\mathcal{N}} \oplus \mathbf{C}^{\rho}_{\mathcal{N}}] \otimes [\tilde{\mathbf{V}}_{\tilde{\mathcal{N}}} \oplus \tilde{\mathbf{C}}^{\tilde{\rho}}_{\tilde{\mathcal{N}}}] = [\mathbf{V}_{\mathcal{N}} \otimes \tilde{\mathbf{V}}_{\tilde{\mathcal{N}}}] \oplus [\mathbf{C}^{\rho}_{\mathcal{N}} \otimes \tilde{\mathbf{C}}^{\tilde{\rho}}_{\tilde{\mathcal{N}}}]. \tag{14}$$

This reflects the BCJ double-copy structure with fundamental matter [15]. As a consequence, although the degrees of freedom of the Yang-Mills theory are not preserved, the twin supergravity theories generated always have the same number of degrees of freedom as they must.

b) In verifying the double-copy relations one must account not only for the content of each theory, but also their symmetries. In particular, for supergravity coupled to matter multiplets one has to trace the Yang-Mills origin of both the R-symmetry and the isotropy group of the matter multiplets. This point is illustrated by the $D = 4$ $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ magic supergravities. Their field content is reproduced by the simple product,

$$\mathbf{V}_2 \otimes [\tilde{A} \oplus n\tilde{\phi}]. \tag{15}$$

for $n = 5, 8, 14, 26$. However, the symmetries generated are those of the generic Jordan sequence $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(2,n)$ and not the magic sequence $\mathrm{USp}(3,\mathbb{R}), \mathrm{SU}(3,3), \mathrm{SO}^{\star}(12), E_{7(-25)}$ [68]. The magic supergravities were double-copy constructed in [18] using a right multiplet given by dimensionally reducing $\mathcal{N} = 0$ Yang-Mills coupled to fundamental fermions in $D = 7, 8, 10, 15$. In the present construction the R-symmetry is always straight-forwardly generated by the left and right R-symmetries, but the isotropy group is more subtle. In particular it can place restrictions on the properties of the left and right gauge group representations, ρ and $\tilde{\rho}$. For example, the $D = 6, \mathcal{N}_{-} = 2$ theory requires ρ to be a real representations of G so as to generate an enhanced flavour symmetry which in turn generates a part of the matter isotropy group.

Before moving on to the $D = 4, (6, 2)$ example, let us briefly return to the "sum of squares rule" noted the above. Adjoint and fundamental representations do not mix in the double-copy prescription:

$$\begin{aligned}
[\mathbf{V}_{\mathcal{N}} \oplus \mathbf{C}^{\rho}_{\mathcal{N}}] \otimes [\tilde{\mathbf{V}}_{\tilde{\mathcal{N}}} \oplus \tilde{\mathbf{C}}^{\tilde{\rho}}_{\tilde{\mathcal{N}}}] &= [\mathbf{V}_{\mathcal{N}} \otimes \tilde{\mathbf{V}}_{\tilde{\mathcal{N}}}] \oplus \cancelto{\varnothing}{[\mathbf{V}_{\mathcal{N}} \otimes \tilde{\mathbf{C}}^{\tilde{\rho}}_{\tilde{\mathcal{N}}}]} \oplus \cancelto{\varnothing}{[\mathbf{C}^{\rho}_{\mathcal{N}} \otimes \tilde{\mathbf{V}}_{\tilde{\mathcal{N}}}]} \oplus [\mathbf{C}^{\rho}_{\mathcal{N}} \otimes \tilde{\mathbf{C}}^{\tilde{\rho}}_{\tilde{\mathcal{N}}}] \\
&= [\mathbf{V}_{\mathcal{N}} \otimes \tilde{\mathbf{V}}_{\tilde{\mathcal{N}}}] \oplus [\mathbf{C}^{\rho}_{\mathcal{N}} \otimes \tilde{\mathbf{C}}^{\tilde{\rho}}_{\tilde{\mathcal{N}}}].
\end{aligned} \tag{16}$$

This is implied by supersymmetry as the cross-terms $\mathbf{V} \otimes \tilde{\mathbf{C}}^{\tilde{\rho}}$ would introduce too many gravitini.

It also follows, with and without supersymmetry, from the structure of colour-kinematic duality for Yang-Mills coupled to fundamental[8] matter and hence the associated double-copy relations, as described in [15]. For adjoint fields colour-kinematic duality is mediated by the Jacobi identity. Of course, the Jacobi identities are just the commutation relations in the adjoint representation, which immediately suggests the appropriate generalisation of colour-kinematic duality to non-adjoint matter [15]. Colour-kinematic duality for fundamental fields is mediated by the commutation relations. In summary, we have

$$[f^a]_c{}^d[f^b]_d{}^e - [f^b]_c{}^d[f^a]_d{}^e = f^{ab}{}_d[f^d]_c{}^e \quad \text{versus} \quad [T^a]_i{}^j[T^b]_j{}^k - [T^b]_i{}^j[T^a]_j{}^k = if^{ab}{}_d[T^d]_i{}^k, \tag{17}$$

where $[f^a]_c{}^d = f^{cad}$, $a = 1, 2, \ldots \dim G$ are the structure constants and $[T^a]_i{}^j$, $i, j = 1, 2, \ldots \dim \rho(G)$ are the generators in the appropriate matter-field representations. Since the colour-kinematic duality applies to triples of graphs with colour factors satisfying either the Jacobi or commutator identities, the fundamental matter multiplets only double-copy amongst themselves and, hence, the product of $\mathbf{C}^{\rho}_{\mathcal{N}}$ with $\tilde{\mathbf{V}}_{\tilde{\mathcal{N}}}$ is trivial.

To be more concrete consider an n-point, L-loop Yang-Mills-matter amplitude, which can be written in terms of trivalent graphs,

$$A_{(1)}{}^L_n = i^L g^{n-2+2L} \sum_i \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D S_i} \frac{n_i c_i}{\prod_{a_i} p^2_{a_i}}. \tag{18}$$

The sum is over all n-point L-loop graphs i with only trivalent vertices. For Yang-Mills coupled to fundamental matter there are two possible classes of trivalent vertex: gluon-gluon-gluon (a, b, c) dressed with a structure constant

[8] Note, the colour-kinematic duality for abelian orbifolds of $D = 4, \mathcal{N} = 4$ super Yang-Mills theory with matter fields in a bi-fundamental representation was studied in [93].

f^{abc} or gluon-fund-antifund $(a,\, i,\, \bar{\jmath})$ dressed with a generator $[T^a]_i{}^j$. The colour factor of graph i is denoted c_i. They are composed of gauge group structure constants f^{abc} and generators $[T^a]_i{}^j$ and can be read-off the graph. The kinematic factor of graph i is denoted n_i. It is a polynomial of Lorentz-invariant contractions of polarisation vectors and momenta and includes any other quantum numbers, which must be specified. The $p^2_{a_i}$ are the propagators for each graph i. S_i is the dimension of the automorphism group of graph i. The set of n-point trivalent graphs can always be organised into triples i, j, k such that the colour factors will obey

$$c_i + c_j + c_k = 0, \tag{19}$$

due to either the Jacobi or commutation relations (17).

For adjoint-valued fields (without fundamental matter) it was proposed in [8] that one can arrange the diagrams to display a remarkable colour-kinematic duality:

$$c_i + c_j + c_k = 0 \;\Rightarrow\; n_i + n_j + n_k = 0 \tag{20}$$

and if $c_i \to -c_i$ under the interchange of two legs then $n_i \to -n_i$. A reorganisation admitting this surprising relationship between colour and kinematic data was shown to exist for all n-point tree-level amplitudes in [9]. The colour-kinematic duality is conjectured to hold, with highly non-trivial evidence [23, 29], at any loop level. This colour-kinematic duality has been extended to include fundamental matter multiplets using commutation relations in place of the Jacobi identity [15].

Assuming one has found a colour-kinematic duality respecting representation of the n-point L-loop (super) Yang-Mills-matter amplitude, mediated by both the Jacobi and commutation relations, the equivalent n-point L-loop (super)gravity amplitude $A_{(2)}{}^L_n$ is obtained by simply replacing each colour factor, c_i, with a second kinematic factor, $\tilde{n}_i$ [8–10, 15]:

$$A_{(1)}{}^L_n = i^L g^{n-2+2L} \sum_i \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D S_i} \frac{n_i c_i}{\prod_{a_i} p^2_{a_i}} \quad \longrightarrow \quad i^L \left(\frac{\kappa}{2}\right)^{n-2+2L} \sum_i \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D S_i} \frac{n_i \tilde{n}_i}{\prod_{a_i} p^2_{a_i}} = A_{(2)}{}^L_n. \tag{21}$$

For pure (super) Yang-Mills at tree-level it was shown in [34] that one can also proceed in the other direction by replacing the kinematic factor n_i by a second colour factor $\tilde{c}_i$ to obtain a spin-0 amplitude $A_{(0)}{}^L_n$ with two independent colour numerators. The result is the n-point tree-level amplitude of a massless bi-adjoint scalar field with cubic interaction,

$$\mathcal{L}_{\text{bi-adj}} = -\frac{1}{2}\partial_\mu \Phi_{a\tilde{a}} \partial^\mu \Phi^{a\tilde{a}} + \frac{\lambda}{6} f_{abc} \tilde{f}_{\tilde{a}\tilde{b}\tilde{c}} \Phi^{a\tilde{a}} \Phi^{b\tilde{b}} \Phi^{c\tilde{c}}, \tag{22}$$

where f_{abc} and $\tilde{f}_{\tilde{a}\tilde{b}\tilde{c}}$ are the structure constants of two independent gauge groups G and $\tilde{G}$. This has been referred to as the zeroth-copy of Yang-Mills [72]. In this sense the Φ^3 theory captures the colour structure of the left and right Yang-Mills factors as well as their propagators, which are common to the equivalent gravitational, gauge and scalar amplitudes.

This bi-adjoint scalar field in fact plays a ubiquitous role in various 'gravity = gauge $\otimes$ gauge' constructions [5, 34, 72, 79]. In the context of amplitudes its appearence is perhaps most clearly expressed in terms of the Cachazo-He-Yuan (CHY) formulae [34]. Remarkably, the spectator field plays a directly analogous role in the double-construction of classical black hole solutions [72–75]. Finally, it was shown to be a crucial element of the linearised *off-shell* dictionary, presented in [5], describing gravitational fields in terms of a convolutive tensor product of left and right Yang-Mills fields,

$$A^a_\mu \circ \Phi_{a\tilde{a}} \circ \tilde{A}^{\tilde{a}}_\nu = g_{\mu\nu} + B_{\mu\nu}. \tag{23}$$

Here $\circ$ denotes a convolutive inner tensor product with respect to the Poincaré group,

$$[f \circ g](x) = \int d^D y f(y) g(x-y). \tag{24}$$

In this context the "spectator" scalar field ensures that the symmetries of the (super) Yang-Mills factors are correctly mapped to those of (super)gravity. The convolution reflects the fact that the amplitude relations are multiplicative in momentum space. It turns out to be essential for reproducing the local symmetries of (super)gravity from those of the two (super) Yang-Mills factors to linear order. The spectator field allows for arbitrary and independent G and $\tilde{G}$ at the level of spacetime fields. Note, the bi-adjoint scalar field also appears by close analogy in the double-copy construction of classical black hole solutions [72–75], although the precise relationship between the two pictures remains an intriguing open question.

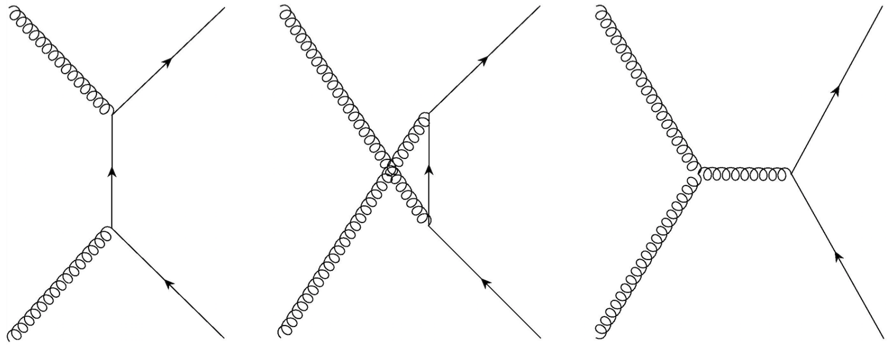

FIG. 1. Tree-level Feynman diagrams for the 4-point gluon-gluon-quark-antiquark interaction.

Now, just as the generalised colour-kinematic duality for adjoint and fundamental multiplets can be used to generate (super)gravity amplitudes with matter couplings, we can once again also proceed in the other direction (at tree-level at least) to generate would-be spin-0 amplitudes. One can then look for the minimal scalar field theory that would produce these amplitudes. The result corresponds to the bi-adjoint scalar theory, but this time coupled to a bi-fundamental scalar field $\Phi_{i\tilde{i}}$ with a cubic interaction term originating from the adj-fund-antifund vertices:

$$\mathcal{L}_{\text{bi-adj-fund}} = -\frac{1}{2}\partial_\mu \Phi_{a\tilde{b}}\partial^\mu \Phi^{a\tilde{b}} - \frac{1}{2}\partial_\mu \Phi_{i\tilde{i}}\partial^\mu \Phi^{i\tilde{i}} + \frac{g}{6}\left(f_{abc}\tilde{f}_{\tilde{a}\tilde{b}\tilde{c}}\Phi^{a\tilde{a}}\Phi^{b\tilde{b}}\Phi^{c\tilde{c}} + i[T^a]_i{}^j[\tilde{T}^{\tilde{a}}]_{\tilde{i}}{}^{\tilde{j}}\Phi_{a\tilde{a}}\Phi^{i\tilde{i}}\Phi_{j\tilde{j}}\right). \tag{25}$$

The defining example is given by the tree-level 4-point gluon-gluon-quark-antiquark interaction. The colour factors of the three Feynman diagrams, given in Figure 1, obey the commutation relation (17). As no gluon 4-point contact term is involved the BCJ representation of the kinematic numerators should, and does, follow directly from the Feynman rules. Using the labelling $(1, \varepsilon_1^\mu(k_1), a), (2, \varepsilon_2^\nu(k_2), b)$ for the two gluons and $(3, v(k_3), {}_i), (4, \bar{u}(k_4), {}^j)$ for the quark-antiquark pair the three Feynman diagrams yield the colour-kinematic numerators:

$$\begin{aligned}
c_1 \times n_1 &= -i[T^a]_i{}^k[T^b]_k{}^j \times \bar{u}\varepsilon_1^\mu\gamma_\mu\gamma^\rho\left(k_4+k_1\right)_\rho\gamma_\nu\varepsilon_2^\nu v\\
c_2 \times n_2 &= -i[T^b]_i{}^k[T^a]_k{}^j \times \bar{u}\varepsilon_2^\nu\gamma_\nu\gamma^\rho\left(k_4+k_2\right)_\rho\gamma_\mu\varepsilon_1^\mu v\\
c_3 \times n_3 &= f^{abc}[T_c]_i{}^j \times \bar{u}\varepsilon_1^\mu\left(\eta_{\mu\rho}(k_1-p)_\nu + \eta_{\rho\nu}(p-k_2)_\mu + \eta_{\mu\nu}(k_2-k_1)_\rho\right)\varepsilon_2^\nu\gamma^\rho v
\end{aligned} \tag{26}$$

where we take all momenta $k_1, \dots k_4$ to be out-going and $p = k_3 + k_4$. Clearly $c_1 - c_2 = c_3$ and going on-shell we find $n_1 - n_2 = n_3$. The corresponding 4-point amplitude with two bi-adjoint and bi-fundamental scalar legs is then given by replacing n_s with a copy $\tilde{c}_s$, which need not carry the same gauge group, as given by double-line Feynman diagrams in Figure 2:

$$\begin{aligned}
c_1 \times \tilde{c}_1 &= -i[T^a]_i{}^k[T^b]_k{}^j \times [\tilde{T}^{\tilde{a}}]_{\tilde{i}}{}^{\tilde{k}}[\tilde{T}^{\tilde{b}}]_{\tilde{k}}{}^{\tilde{j}}\\
c_2 \times \tilde{c}_2 &= -i[T^b]_i{}^k[T^a]_k{}^j \times [\tilde{T}^{\tilde{b}}]_{\tilde{i}}{}^{\tilde{k}}[\tilde{T}^{\tilde{a}}]_{\tilde{k}}{}^{\tilde{j}}\\
c_3 \times \tilde{c}_3 &= f^{abc}[T_c]_i{}^j \times \tilde{f}^{\tilde{a}\tilde{b}\tilde{c}}[\tilde{T}_{\tilde{c}}]_{\tilde{i}}{}^{\tilde{j}}
\end{aligned} \tag{27}$$

As for the double-copy, the denominators are the same for both amplitudes. Note, given a Feynman diagram representation, such as we have here, the dictionary is quite intuitive. For each vertex, shorn of external states, the replacement rules are simply

$$\gamma^\mu{}_\alpha{}^\beta \to [\tilde{T}^{\tilde{a}}]_{\tilde{i}}{}^{\tilde{j}} \tag{28}$$

and

$$\eta_{\mu_1\mu_2}(k_1-k_2)_{\mu_3} + \eta_{\mu_2\mu_3}(k_2-k_3)_{\mu_1} + \eta_{\mu_3\mu_1}(k_3-k_1)_{\mu_2} \to \tilde{f}^{\tilde{a}_1\tilde{a}_2\tilde{a}_3}. \tag{29}$$

All contractions of gauge group indices amongst vertices are dictated by the contractions of the corresponding kinematic indices by propagators. Note, we can accommodate a broader class of double-copy constructions by also including a fundamental-antifundamental scalar $\Phi^i{}_{\tilde{i}}$, allowing for distinct colour structures in the two factors mirroring the product of distinct kinematic structures in the double-copy for gravitational amplitudes. This will be

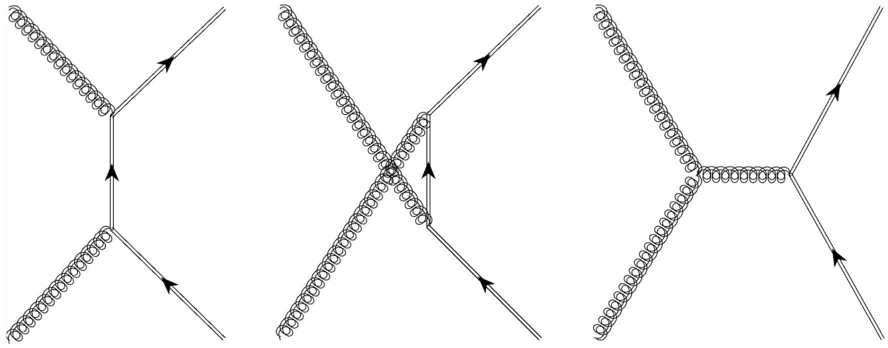

FIG. 2. Double-line tree-level Feynman diagrams for the 4-point (bi-adj.)-(bi-adj.)-(bi-fund.)-(bi-fund.) interaction for the scalar theory in (25). The curly (straight) double-lines represent the bi-adjoint (bi-fundamental) representation of the global $G \times \tilde{G}$ symmetry. Each diagram is the double-copy of the corresponding gluon-quark diagram shawn of its kinematic data.

developed in future work. The scalar field theory can also be further generalised to accommodate structural features such as quark flavours. A 5-point example, following [15], illustrating this point is given in appendix B.

Most importantly for the present work is that the bi-fundamental scalar is consistent with spacetime field dictionary of [5]. In particular, we can generalise (23) to include fundamental fields in the left and right multiplets by introducing a block-diagonal spectator field Φ with bi-adjoint and bi-fundamental sectors,

$$\Phi = \begin{pmatrix} \Phi^{a\tilde{b}} & 0 \\ 0 & \Phi^{i\tilde{j}} \end{pmatrix}. \tag{30}$$

The Lorentz covariant position-space dictionary then correctly captures the sum-of-squares rule:

$$[\mathbf{V}_{\mathcal{N}} \oplus \mathbf{C}^{\rho}_{\mathcal{N}}] \star [\tilde{\mathbf{V}}_{\tilde{\mathcal{N}}} \oplus \tilde{\mathbf{C}}^{\tilde{\rho}}_{\tilde{\mathcal{N}}}] = [\mathbf{V}_{\mathcal{N}} \oplus \mathbf{C}^{\rho}_{\mathcal{N}}] \circ \Phi \circ [\tilde{\mathbf{V}}_{\tilde{\mathcal{N}}} \oplus \tilde{\mathbf{C}}^{\tilde{\rho}}_{\tilde{\mathcal{N}}}] = \mathbf{V}^{a}_{\mathcal{N}} \circ \Phi_{a\tilde{a}} \circ \tilde{\mathbf{V}}^{\tilde{a}}_{\tilde{\mathcal{N}}} \oplus \mathbf{C}^{i}_{\mathcal{N}} \circ \Phi_{i\tilde{i}} \circ \tilde{\mathbf{C}}^{\tilde{i}}_{\tilde{\mathcal{N}}}. \tag{31}$$

Crucially, the symmetries of the Yang-Mills-matter factors are correctly mapped to the global and local (super)gravity symmetries via $\star$ and Φ. This allows us to establish the corresponding supergravity theory through the structure of the Yang-Mills symmetries alone, assuming that the gravitational scalar fields parametrise symmetric spaces.

A. Example: the $D = 4, (6, 2)$ twin theories

To make this procedure concrete let us consider in detail the prototypical example: $D = 4, \mathcal{N}_+ = 6$ pure supergravity and its twin, the magic $D = 4, \mathcal{N}_- = 2$ supergravity coupled to 15 vector multiplets.

The $D = 4, \mathcal{N} = 6$ supergravity theory is unique and determined by supersymmetry. The multiplet consists of

$$\mathbf{G}_6 = \{g_{\mu\nu}, 16A_\mu, 30\phi; 6\Psi_\mu, 26\chi\}. \tag{32}$$

The R-symmetry algebra is $\mathfrak{u}(6)$ under which the on-shell helicity states transform as:

$$\begin{array}{rcccl} & & \mathfrak{so}(2) & \mathfrak{u}(6) & \\ & | & 2 & \mathbf{1}_0 & \rangle \\ Q & | & \frac{3}{2} & \mathbf{6}_1 & \rangle \\ \wedge^2 Q & | & 1 & \mathbf{15}_2 & \rangle \\ \wedge^3 Q & | & \frac{1}{2} & \mathbf{20}_3 & \rangle \\ \wedge^4 Q & | & 0 & \overline{\mathbf{15}}_4 & \rangle \\ \wedge^5 Q & | & -\frac{1}{2} & \overline{\mathbf{6}}_5 & \rangle \\ \wedge^6 Q & | & -1 & \mathbf{1}_6 & \rangle \end{array} \qquad \begin{array}{rcccl} & & \mathfrak{so}(2) & \mathfrak{u}(6) & \\ & | & 1 & \mathbf{1}_{-6} & \rangle \\ Q & | & \frac{1}{2} & \mathbf{6}_{-5} & \rangle \\ \wedge^2 Q & | & 0 & \mathbf{15}_{-4} & \rangle \\ \wedge^3 Q & | & -\frac{1}{2} & \mathbf{20}_{-3} & \rangle \\ \wedge^4 Q & | & -1 & \overline{\mathbf{15}}_{-2} & \rangle \\ \wedge^5 Q & | & -\frac{3}{2} & \overline{\mathbf{6}}_{-1} & \rangle \\ \wedge^6 Q & | & -2 & \mathbf{1}_0 & \rangle \end{array} \tag{33}$$

The non-compact global symmetry of the equations of motion is $\mathrm{SO}^\star(12)$. The 15+15 scalars parametrise the coset manifold $\mathrm{SO}^\star(12)/\mathrm{U}(6)$,

$$\begin{array}{lcl} \mathfrak{so}^\star(12) & \supset & \mathfrak{u}(1) \oplus \mathfrak{su}(6); \\ \mathbf{66} & \to & [\mathbf{1} + \mathbf{35}]_0 + \mathbf{15}_{-4} + \overline{\mathbf{15}}_4. \end{array} \tag{34}$$

The 16 Maxwell field strengths and their duals transform as the **32** of SO*(12),

$$\begin{array}{lcl} \mathfrak{so}^\star(12) & \supset & \mathfrak{u}(1)\oplus\mathfrak{su}(6); \\ \mathbf{32} & \to & \mathbf{1}_6+\mathbf{1}_{-6}+\mathbf{15}_2+\overline{\mathbf{15}}_{-2}. \end{array} \tag{35}$$

The parent theory is $\mathcal{N}_+ + \mathcal{N}_- = 8$ supergravity with on-shell helicity states,

$$\begin{array}{rcccl} & & \mathfrak{so}(2) & \mathfrak{su}(8) & \\ & | & 2 & \mathbf{1} & \rangle \\ Q & | & \frac{3}{2} & \mathbf{8} & \rangle \\ \wedge^2 Q & | & 1 & \mathbf{28} & \rangle \\ \wedge^3 Q & | & \frac{1}{2} & \mathbf{56} & \rangle \\ \wedge^4 Q & | & 0 & \mathbf{70} & \rangle \\ \wedge^5 Q & | & -\frac{1}{2} & \overline{\mathbf{56}} & \rangle \\ \wedge^6 Q & | & -1 & \overline{\mathbf{28}} & \rangle \\ \wedge^7 Q & | & -\frac{3}{2} & \overline{\mathbf{8}} & \rangle \\ \wedge^8 Q & | & -2 & \mathbf{1} & \rangle \end{array} \tag{36}$$

As a truncation the $\mathcal{N}_+ = 6$ theory is obtained by decomposing with respect to $\mathfrak{su}(8) \supset \mathfrak{su}(6)\times\mathfrak{su}(2)\times\mathfrak{u}(1)$ and discarding the non-trivial $\mathfrak{su}(2)$ representations,

$$\begin{array}{lcl} \mathbf{8} & \to & (\mathbf{1},\mathbf{2})_{\text{-}3}+(\mathbf{6},\mathbf{1})_1 \\ \mathbf{28} & \to & (\mathbf{1},\mathbf{1})_{\text{-}6}+(\mathbf{6},\mathbf{2})_{\text{-}2}+(\mathbf{15},\mathbf{1})_2 \\ \mathbf{56} & \to & (\mathbf{6},\mathbf{1})_{\text{-}5}+(\mathbf{15},\mathbf{2})_{\text{-}1}+(\mathbf{20},\mathbf{1})_3 \\ \mathbf{70} & \to & (\mathbf{15},\mathbf{1})_{\text{-}4}+\left(\overline{\mathbf{15}},\mathbf{1}\right)_4+(\mathbf{20},\mathbf{2})_0 \\ \overline{\mathbf{56}} & \to & (\bar{\mathbf{6}},\mathbf{1})_5+\left(\overline{\mathbf{15}},\mathbf{2}\right)_1+(\mathbf{20},\mathbf{1})_{\text{-}3} \\ \overline{\mathbf{28}} & \to & (\mathbf{1},\mathbf{1})_6+(\bar{\mathbf{6}},\mathbf{2})_2+\left(\overline{\mathbf{15}},\mathbf{1}\right)_{\text{-}2} \\ \overline{\mathbf{8}} & \to & (\mathbf{1},\mathbf{2})_3+(\bar{\mathbf{6}},\mathbf{1})_{\text{-}1} \end{array} \tag{37}$$

From the perspective of Yang-Mills squared we have the unique product,

$$\mathbf{V}_4\otimes\tilde{\mathbf{V}}_4=\mathbf{G}_8 \tag{38}$$

and the $\mathcal{N}_+ = 6$ multiplet is the product of $\mathcal{N}=2$ and $\mathcal{N}=4$ vector multiplets. The above truncation from $\mathcal{N}=8$ is effected by decomposing one of the $\mathcal{N}=4$ vector multiplets (we choose the left) into an $\mathcal{N}=2$ vector-multiplet plus hyper-multiplet,

$$\begin{array}{rcccl} & & \mathfrak{so}(2) & \mathfrak{su}(4) & \\ & | & 1 & \mathbf{1} & \rangle \\ Q & | & \frac{1}{2} & \mathbf{4} & \rangle \\ \wedge^2 Q & | & 0 & \mathbf{6} & \rangle \\ \wedge^3 Q & | & -\frac{1}{2} & \overline{\mathbf{4}} & \rangle \\ \wedge^4 Q & | & -1 & \mathbf{1} & \rangle \end{array} \longrightarrow \begin{array}{rcccl} & & \mathfrak{so}(2) & \mathfrak{u}(2)\oplus\mathfrak{su}(2) & \\ & | & 1 & (\mathbf{1},\mathbf{1})_0 & \rangle \\ Q & | & \frac{1}{2} & (\mathbf{2},\mathbf{1})_1 & \rangle \\ \wedge^2 Q & | & 0 & (\mathbf{1},\mathbf{1})_2 & \rangle \\ & & & & \\ & | & 0 & (\mathbf{1},\mathbf{1})_{-2} & \rangle \\ Q & | & -\frac{1}{2} & (\mathbf{2},\mathbf{1})_{-1} & \rangle \\ \wedge^2 Q & | & -1 & (\mathbf{1},\mathbf{1})_0 & \rangle \end{array} \oplus \begin{array}{rcccl} & & \mathfrak{so}(2) & \mathfrak{u}(2)\oplus\mathfrak{su}(2) & \\ & | & \frac{1}{2} & (\mathbf{1},\mathbf{2})_{-1} & \rangle \\ Q & | & 0 & (\mathbf{2},\mathbf{2})_0 & \rangle \\ \wedge^2 Q & | & -\frac{1}{2} & (\mathbf{1},\mathbf{2})_1 & \rangle \end{array} \tag{39}$$

where $\mathfrak{su}(4)_R \supset \mathfrak{u}(2)_R\oplus\mathfrak{su}(2)\oplus\mathfrak{u}(1)$. Rather than truncate $\mathbf{H}_2$ we replace it with another hyper-multiplet in a fundamental representation ρ of the gauge group,

$$\mathbf{V}_4\otimes\tilde{\mathbf{V}}_4=\mathbf{G}_8\longrightarrow[\mathbf{V}_2\oplus\mathbf{H}_2^\rho]\otimes\tilde{\mathbf{V}}_4=\mathbf{G}_6, \tag{40}$$

as indicated by the superscript. This reduces the left supersymmetry from $\mathcal{N}=4$ to $\mathcal{N}'=2$. To preserve the $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ symmetry ρ must be a real representation. The second $\mathfrak{su}(2)$ factor is an enhanced flavour symmetry that is only present for real representations of the gauge group. This is a special case of the enhanced $\mathfrak{sp}(n)$ or $\mathfrak{so}(n)$ flavour symmetry enjoyed by n hypermultiplets in a real or pseudo-real gauge group representation, respectively. As $\mathbf{H}_2^{\rho}$ does not 'talk' to the right adjoint valued multiplet $\tilde{\mathbf{V}}_4$, from the perspective of squaring it is effectively truncated. Since $\mathbf{V}_2$ is a singlet under the $\mathfrak{su}(2)$ flavour it plays no role here either; this reflects the fact that $\mathcal{N}=6$ supergravity does not admit matter couplings and there is no corresponding isotropy group. The $\mathfrak{u}(6)$ of R-symmetry is, roughly speaking, generated by the left and right $\mathfrak{u}(2)$ and $\mathfrak{su}(4)$ R-symmetries. Explicitly, we have

$$
\begin{array}{l|lllll}
\mathcal{N}'=2\backslash\tilde{\mathcal{N}}=4 & |1;\mathbf{1}\rangle & |\tfrac{1}{2};\mathbf{4}\rangle & |0;\mathbf{6}\rangle & |\bar{\tfrac{1}{2}};\overline{\mathbf{4}}\rangle & |\bar{1};\mathbf{1}\rangle \\
\hline
|1;(\mathbf{1},\mathbf{1})_0\rangle & |2;(\mathbf{1},\mathbf{1},\mathbf{1})_{00}\rangle & |\tfrac{3}{2};(\mathbf{1},\mathbf{1},\mathbf{4})_{0\frac{1}{2}}\rangle & |1;(\mathbf{1},\mathbf{1},\mathbf{6})_{01}\rangle & |\tfrac{1}{2};(\mathbf{1},\mathbf{1},\overline{\mathbf{4}})_{0\frac{3}{2}}\rangle & |0;(\mathbf{1},\mathbf{1},\mathbf{1})_{02}\rangle \\
|\tfrac{1}{2};(\mathbf{2},\mathbf{1})_1\rangle & |\tfrac{3}{2};(\mathbf{2},\mathbf{1},\mathbf{1})_{1\bar{\frac{1}{2}}}\rangle & |1;(\mathbf{2},\mathbf{1},\mathbf{4})_{10}\rangle & |\tfrac{1}{2};(\mathbf{2},\mathbf{1},\mathbf{6})_{1\frac{1}{2}}\rangle & |0;(\mathbf{2},\mathbf{1},\overline{\mathbf{4}})_{11}\rangle & |\bar{\tfrac{1}{2}};(\mathbf{2},\mathbf{1},\mathbf{1})_{1\frac{3}{2}}\rangle \\
|\tfrac{1}{2};(\mathbf{1},\mathbf{2})_{\bar{1}}\rangle_\rho & & & & & \\
|0;(\mathbf{1},\mathbf{1})_2\rangle & |1;(\mathbf{1},\mathbf{1},\mathbf{1})_{2\bar{1}}\rangle & |\tfrac{1}{2};(\mathbf{1},\mathbf{1},\mathbf{4})_{2\bar{\frac{1}{2}}}\rangle & |0;(\mathbf{1},\mathbf{1},\mathbf{6})_{20}\rangle & |\bar{\tfrac{1}{2}};(\mathbf{1},\mathbf{1},\overline{\mathbf{4}})_{2\frac{1}{2}}\rangle & |\bar{1};(\mathbf{1},\mathbf{1},\mathbf{1})_{21}\rangle \\
|0;(\mathbf{2},\mathbf{2})_0\rangle_\rho & & & & & \\
|0;(\mathbf{1},\mathbf{1})_{\bar{2}}\rangle & |1;(\mathbf{1},\mathbf{1},\mathbf{1})_{\bar{2}\bar{1}}\rangle & |\tfrac{1}{2};(\mathbf{1},\mathbf{1},\mathbf{4})_{\bar{2}\bar{\frac{1}{2}}}\rangle & |0;(\mathbf{1},\mathbf{1},\mathbf{6})_{\bar{2}0}\rangle & |\bar{\tfrac{1}{2}};(\mathbf{1},\mathbf{1},\overline{\mathbf{4}})_{\bar{2}\frac{1}{2}}\rangle & |\bar{1};(\mathbf{1},\mathbf{1},\mathbf{1})_{\bar{2}}\rangle \\
|\tfrac{1}{2};(\mathbf{1},\mathbf{1},\mathbf{2})_1\rangle_\rho & & & & & \\
|\bar{\tfrac{1}{2}};(\mathbf{2},\mathbf{1})_{\bar{1}}\rangle & |\tfrac{1}{2};(\mathbf{2},\mathbf{1},\mathbf{1})_{\bar{1}\frac{3}{2}}\rangle & |0;(\mathbf{2},\mathbf{1},\mathbf{4})_{\bar{1}\bar{1}}\rangle & |\bar{\tfrac{1}{2}};(\mathbf{2},\mathbf{1},\mathbf{6})_{\bar{1}\frac{1}{2}}\rangle & |\bar{1};(\mathbf{2},\mathbf{1},\overline{\mathbf{4}})_{\bar{1}0}\rangle & |\bar{\tfrac{3}{2}};(\mathbf{2},\mathbf{1},\mathbf{1})_{\bar{1}\frac{1}{2}}\rangle \\
|\bar{1};(\mathbf{1},\mathbf{1})_0\rangle & |0;(\mathbf{1},\mathbf{1},\mathbf{1})_{0\bar{2}}\rangle & |\bar{\tfrac{1}{2}};(\mathbf{1},\mathbf{1},\mathbf{4})_{0\frac{3}{2}}\rangle & |\bar{1};(\mathbf{1},\mathbf{1},\mathbf{6})_{0\bar{1}}\rangle & |\bar{\tfrac{3}{2}};(\mathbf{1},\mathbf{1},\overline{\mathbf{4}})_{0\frac{1}{2}}\rangle & |\bar{2};(\mathbf{1},\mathbf{1},\mathbf{1})_{00}\rangle
\end{array}
\tag{41}
$$

where for notational clarity we have used $\bar{\lambda}\equiv-\lambda$ to denote negative helicities. In the above we have included the effectively trivial states belonging to the fundamental hypermultiplet (as indicated by the ρ subscript) to illustrate how the truncation of the $\mathcal{N}=8$ theory is effected. Here the left $\mathcal{N}=2$ super Yang-Mills states carry $\mathfrak{so}(2)_l\oplus\mathfrak{u}(2)_L\oplus\mathfrak{su}(2)_L$ spacetime little group, R-symmetry and flavour representations and the right $\mathcal{N}_R=4$ states carry $\mathfrak{so}(2)_r\oplus\mathfrak{su}(4)_R$ spacetime little group and R-symmetry representations. The $\mathcal{N}=6$ supergravity states carry $\mathfrak{so}(2)_{st}\oplus[\mathfrak{u}(2)_L\oplus\mathfrak{su}(2)_L]\oplus[\mathfrak{su}(4)_R]\oplus\mathfrak{u}(1)$ representations, where the spacetime helicity group $\mathfrak{so}(2)_{st}$ and the additional $\mathfrak{u}(1)$ factor are given by the sum and difference of the $\mathfrak{so}(2)_l$ and $\mathfrak{so}(2)_r$ generators, respectively. The charges carried by the extra $\mathfrak{u}(1)$ are given by the second subscript. Following [70] the $[\mathfrak{u}(2)_L]\oplus[\mathfrak{su}(4)_R]\oplus\mathfrak{u}(1)$ generators are completed to the $\mathcal{N}=6$ R-symmetry algebras $\mathfrak{u}(6)$. All states are trivial under $\mathfrak{su}(2)_L$, which drops out of the equations.

Before the states can be assembled into the corresponding irreducible $\mathcal{N}=6$ multiplet we have to take a linear combination of the $\mathfrak{u}(1)_L$ and $\mathfrak{u}(1)$ generators,

$$
\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} h_L \\ h \end{pmatrix}. \tag{42}
$$

We then reproduce the states as given in (33), as can be seen by comparing (41) with the following decompositions:

$$
\begin{array}{lcl}
\mathfrak{su}(6) & \supset & \mathfrak{su}(2)\oplus\mathfrak{su}(4)\oplus\mathfrak{u}(1) \\
 & & \\
\mathbf{6} & \to & (\mathbf{2},\mathbf{1})_{-2}+(\mathbf{1},\mathbf{4})_1 \\
\mathbf{15} & \to & (\mathbf{1},\mathbf{1})_{-4}+(\mathbf{2},\mathbf{4})_{-1}+(\mathbf{1},\mathbf{6})_2 \\
\mathbf{20} & \to & (\mathbf{1},\mathbf{4})_{-3}+(\mathbf{1},\overline{\mathbf{4}})_3+(\mathbf{2},\mathbf{6})_0 \\
\overline{\mathbf{15}} & \to & (\mathbf{1},\mathbf{1})_4+(\mathbf{2},\overline{\mathbf{4}})_1+(\mathbf{1},\mathbf{6})_{-2} \\
\overline{\mathbf{6}} & \to & (\mathbf{2},\mathbf{1})_2+(\mathbf{1},\overline{\mathbf{4}},)_{-1}
\end{array}
\tag{43}
$$

Its little twin theory is the magic $\mathcal{N}=2$ supergravity coupled to 15 vector multiplets based on the Jordan algebra of 3×3 Hermitian quaternionic matrices $\mathfrak{J}_3(\mathbb{H})$, with content:

$$
\mathbf{G}_2\oplus15\mathbf{V}_2=\{g_{\mu\nu},A_\mu;2\Psi_\mu\}\oplus15\{A_\mu,2\phi;2\chi\}. \tag{44}
$$

The R-symmetry algebra is $\mathfrak{u}(2)_R$ and the isotropy algebra is $\mathfrak{su}(6)$ under which the on-shell helicity states transform as:

$$
\begin{array}{rccc}
 & \mathfrak{so}(2) & \mathfrak{u}(2)_R \oplus \mathfrak{su}(6) & \\
 | & 2 & (\mathbf{1},\mathbf{1})_0 & \rangle \\
Q\;| & \frac{3}{2} & (\mathbf{2},\mathbf{1})_1 & \rangle \\
\wedge^2 Q\;| & 1 & (\mathbf{1},\mathbf{1})_2 & \rangle \\
 & & & \\
 | & -1 & (\mathbf{1},\mathbf{1})_{-2} & \rangle \\
Q\;| & -\frac{3}{2} & (\mathbf{2},\mathbf{1})_{-1} & \rangle \\
\wedge^2 Q\;| & -2 & (\mathbf{1},\mathbf{1})_0 & \rangle
\end{array}
\quad\oplus\quad
\begin{array}{rccc}
 & \mathfrak{so}(2) & \mathfrak{u}(2)_R \oplus \mathfrak{su}(6) & \\
 | & 1 & (\mathbf{1},\overline{\mathbf{15}})_0 & \rangle \\
Q\;| & \frac{1}{2} & (\mathbf{2},\overline{\mathbf{15}})_1 & \rangle \\
\wedge^2 Q\;| & 0 & (\mathbf{1},\overline{\mathbf{15}})_2 & \rangle \\
 & & & \\
 | & 0 & (\mathbf{1},\mathbf{15})_{-2} & \rangle \\
Q\;| & -\frac{1}{2} & (\mathbf{2},\mathbf{15})_{-1} & \rangle \\
\wedge^2 Q\;| & -1 & (\mathbf{1},\mathbf{15})_0 & \rangle
\end{array}
\tag{45}
$$

The 30 scalars parametrise the coset manifold

$$
\frac{\mathrm{SU}(2)}{\mathrm{SU}(2)} \times \frac{\mathrm{SO}^\star(12)}{\mathrm{U}(6)} \cong \frac{\mathrm{SO}^\star(12)}{\mathrm{U}(6)},
\tag{46}
$$

where the U(1) of the R-symmetry has been “gauged” such that

$$
\begin{array}{lcl}
\mathfrak{so}^\star(12) & \supset & \mathfrak{u}(1) \oplus \mathfrak{su}(6); \\
\mathbf{66} & \to & [\mathbf{1}+\mathbf{35}]_0 + \mathbf{15}_{-4} + \overline{\mathbf{15}}_4 .
\end{array}
\tag{47}
$$

The non-compact global symmetry of the equations of motion is $\mathrm{SO}^\star(12)$, under which the 16 Maxwell field strengths and their duals comprise the $\mathbf{32}$ spinor representation,

$$
\begin{array}{lcl}
\mathfrak{so}^\star(12) & \supset & \mathfrak{u}(1) \oplus \mathfrak{su}(6); \\
\mathbf{32} & \to & \mathbf{1}_6 + \mathbf{1}_{-6} + \mathbf{15}_2 + \overline{\mathbf{15}}_{-2} .
\end{array}
\tag{48}
$$

As described in the original treatment of the magic supergravities [90, 94, 95], the 15 potentials and their duals can be regarded as elements of the Jordan algebra of 3×3 Hermitian matrices defined over the quaternions, $\mathfrak{J}_3(\mathbb{H})$, and its dual with respect to the bilinear Jordan trace form, $\mathfrak{J}_3(\mathbb{H})^* \cong \mathfrak{J}_3(\mathbb{H})$. Combined with the graviphoton and its dual these can be assembled into a 32-dimensional Freudenthal triple system and the pair $(\mathrm{SO}^\star(12), \mathbf{32})$ constitutes a group of type E_7.

To generate the magic $\mathcal{N}_- = 2$ theory we similarly decompose the right $\tilde{\mathcal{N}} = 4$ multiplet into $\tilde{\mathcal{N}}' = 0$ multiplets,

$$
\mathbf{V}_4 = \{\tilde{A}, \tilde{\phi}_{[\alpha\beta]}; \tilde{\chi}_\alpha^{\tilde{\rho}}\}
\tag{49}
$$

where $\alpha, \beta = 1, \ldots 4$ are indices of the fundamental of the R-symmetry remnant $\mathfrak{su}(4)$. Here we have replaced the *adjoint-valued* $\tilde{\chi}_\alpha$ by a *fundamental-valued* spinor $\tilde{\chi}_\alpha^{\tilde{\rho}}$ reducing the degree of supersymmetry. We then have the twin truncations of the parent $\mathscr{N} = 8$ theory:

$$
\begin{tikzcd}
 & \begin{array}{c}\mathscr{N}=8 \text{ Parent supergravity}\\ \mathbf{G}_8\end{array} \arrow[d, "\text{Yang-Mills factors}"] & \\
 & \mathbf{V}_4 \otimes \tilde{\mathbf{V}}_4 \arrow[dl] \arrow[dr] & \\
{[\mathbf{V}_2 \oplus \mathbf{H}_2^\rho] \otimes \tilde{\mathbf{V}}_4} \arrow[rr, leftrightarrow, dashed, "\text{twin relation}"'] \arrow[d] & & {[\mathbf{V}_2 \oplus \mathbf{H}_2^\rho] \otimes [\tilde{A} \oplus \tilde{\chi}_\alpha^{\tilde{\rho}} \oplus \tilde{\phi}_{[\alpha\beta]}]} \arrow[d] \\
\mathbf{V}_2 \otimes \tilde{\mathbf{V}}_4 \arrow[d] & & {[\mathbf{V}_2 \otimes (\tilde{A} \oplus \tilde{\phi}_{[\alpha\beta]})] \oplus [\mathbf{H}_2^\rho \otimes \tilde{\chi}_\alpha^{\tilde{\rho}}]} \arrow[d] \\
\begin{array}{c}\mathcal{N}_+ = 6 \text{ supergravity}\\ \mathbf{G}_6\end{array} & & \begin{array}{c}\mathcal{N}_- = 2 \text{ magic supergravity}\\ \mathbf{G}_2 \oplus \mathbf{V}_2 \oplus 2\mathbf{V}_{2\alpha} \oplus \mathbf{V}_{2[\alpha\beta]}\end{array}
\end{tikzcd}
\tag{50}
$$

Explicitly, we have the complementary truncation, cf. (41), of $\mathscr{N}=8$ supergravity in terms of the left and right helicity states:

$$
\begin{array}{c|ccccc}
\mathcal{N}'=2\backslash\tilde{\mathcal{N}}'=0 & |1;\mathbf{1}\rangle & |\tfrac{1}{2};\mathbf{4}\rangle_{\tilde\rho} & |0;\mathbf{6}\rangle & |\tfrac{\bar1}{2};\overline{\mathbf{4}}\rangle_{\tilde\rho} & |\bar1;\mathbf{1}\rangle \\
\hline
|1;(\mathbf{1},\mathbf{1})_0\rangle & |2;(\mathbf{1},\mathbf{1},\mathbf{1})_{00}\rangle & & |1;(\mathbf{1},\mathbf{1},\mathbf{6})_{01}\rangle & & |0;(\mathbf{1},\mathbf{1},\mathbf{1})_{02}\rangle \\
|\tfrac{1}{2};(\mathbf{2},\mathbf{1})_1\rangle & |\tfrac{3}{2};(\mathbf{2},\mathbf{1},\mathbf{1})_{1\frac{\bar1}{2}}\rangle & & |\tfrac{1}{2};(\mathbf{2},\mathbf{1},\mathbf{6})_{1\frac{1}{2}}\rangle & & |\tfrac{\bar1}{2};(\mathbf{2},\mathbf{1},\mathbf{1})_{1\frac{3}{2}}\rangle \\
|0;(\mathbf{1},\mathbf{1})_2\rangle & |1;(\mathbf{1},\mathbf{1},\mathbf{1})_{2\bar1}\rangle & & |0;(\mathbf{1},\mathbf{1},\mathbf{6})_{20}\rangle & & |\bar1;(\mathbf{1},\mathbf{1},\mathbf{1})_{21}\rangle \\
|0;(\mathbf{1},\mathbf{1})_{\bar2}\rangle & |1;(\mathbf{1},\mathbf{1},\mathbf{1})_{\bar2\bar1}\rangle & & |0;(\mathbf{1},\mathbf{1},\mathbf{6})_{\bar20}\rangle & & |\bar1;(\mathbf{1},\mathbf{1},\mathbf{1})_{\bar21}\rangle \\
|\tfrac{\bar1}{2};(\mathbf{2},\mathbf{1})_{\bar1}\rangle & |\tfrac{1}{2};(\mathbf{2},\mathbf{1},\mathbf{1})_{\bar1\frac{3}{2}}\rangle & & |\tfrac{\bar1}{2};(\mathbf{2},\mathbf{1},\mathbf{6})_{\bar1\frac{\bar1}{2}}\rangle & & |\tfrac{\bar3}{2};(\mathbf{2},\mathbf{1},\mathbf{1})_{\bar1\frac{1}{2}}\rangle \\
|\bar1;(\mathbf{1},\mathbf{1})_0\rangle & |0;(\mathbf{1},\mathbf{1},\mathbf{1})_{0\bar2}\rangle & & |\bar1;(\mathbf{1},\mathbf{1},\mathbf{6})_{0\bar1}\rangle & & |\bar2;(\mathbf{1},\mathbf{1},\mathbf{1})_{00}\rangle \\
 & & & & & \\
|\tfrac{1}{2};(\mathbf{1},\mathbf{2})_{\bar1}\rangle_\rho & & |1;(\mathbf{1},\mathbf{2},\mathbf{4})_{10}\rangle & & |0;(\mathbf{1},\mathbf{2},\overline{\mathbf{4}})_{11}\rangle & \\
|0;(\mathbf{2},\mathbf{2})_0\rangle_\rho & & |\tfrac{1}{2};(\mathbf{2},\mathbf{2},\mathbf{4})_{\bar2\frac{\bar1}{2}}\rangle & & |\tfrac{\bar1}{2};(\mathbf{2},\mathbf{2},\overline{\mathbf{4}})_{\bar2\frac{1}{2}}\rangle & \\
|\tfrac{\bar1}{2};(\mathbf{1},\mathbf{2})_1\rangle_\rho & & |0;(\mathbf{1},\mathbf{2},\mathbf{4})_{\bar1\bar1}\rangle & & |\bar1;(\mathbf{1},\mathbf{2},\overline{\mathbf{4}})_{\bar10}\rangle &
\end{array}
\tag{51}
$$

Here the left $\mathcal{N}'=2$ multiplet states carry $\mathfrak{so}(2)_l$ spacetime little group and $\mathfrak{u}(2)_L\oplus\mathfrak{su}(2)_L$ R-symmetry plus enhanced flavour representations. The right $\tilde{\mathcal{N}}'=0$ multiplet states carry $\mathfrak{so}(2)_r$ spacetime little group and $\mathfrak{su}(4)_R$ representations, where the $\mathfrak{su}(4)_R$ can be regarded as the remnant of the $\tilde{\mathcal{N}}=4$ R-symmetry. The $\mathcal{N}_-=2$ supergravity and vector multiplet states carry $\mathfrak{so}(2)_{st}\oplus\mathfrak{u}(2)_L\oplus[\mathfrak{su}(2)_L\oplus\mathfrak{su}(4)_R]\oplus\mathfrak{u}(1)$ representations, where the spacetime helicity group $\mathfrak{so}(2)_{st}$ and the additional $\mathfrak{u}(1)$ are given by the sum and difference of the $\mathfrak{so}(2)_l$ and $\mathfrak{so}(2)_r$ generators, respectively. The charges carried by the extra $\mathfrak{u}(1)$ are given by the second subscript.

The $\mathfrak{u}(2)_L$ R-symmetry of the of the left multiplet carries over as the R-symmetry of the gravity plus vector multiplets. The additional $\mathfrak{u}(1)$ and enhanced flavour $\mathfrak{su}(2)_L$ together with the $\mathfrak{su}(4)_R$ R-symmetry remnant of the right multiplet are enhanced to provide the $\mathfrak{su}(6)$ isotropy group. As for the $\mathcal{N}_+=6$ twin we must take the same linear combination of $\mathfrak{u}(1)$ generators to organise the states into $\mathfrak{su}(6)$ representations. Note, the R-symmetry representations simply go along for the ride. Using

$$
\begin{aligned}
\mathfrak{su}(6) &\supset \mathfrak{su}(2)\oplus\mathfrak{su}(4)\oplus\mathfrak{u}(1)\\
\mathbf{15} &\to (\mathbf{1},\mathbf{1})_{-4}+(\mathbf{2},\mathbf{4})_{-1}+(\mathbf{1},\mathbf{6})_{2}\\
\overline{\mathbf{15}} &\to (\mathbf{1},\mathbf{1})_{4}+(\mathbf{2},\overline{\mathbf{4}})_{1}+(\mathbf{1},\mathbf{6})_{-2}
\end{aligned}
\tag{52}
$$

we find that the spectrum of (45) is reproduced.

This summarises the origin of the $D=4,(6,2)$ twins from the perspective of Yang-Mills squared. It should be noted that the magic supergravity described here was previously double-copy constructed in [18] as the the product of an $\mathcal{N}=2$ vector multiplet and $\mathcal{N}=0$ vector potential coupled to six adjoint scalars and 8 pseudo-real fermions in the $(\mathbf{2},\mathbf{8})$ of $\mathfrak{su}(2)\oplus\mathfrak{su}(4)$.

B. Summary: the pyramid twins in $D=3,4,5,6$

The remaining examples in $D=3,4,5$ follow precisely the same pattern and we accordingly omit the details. The results are summarised in Table IV, Table V, Table VI and Table VII. At this stage some comments are in order. First, the twin relation generates new double-copy constructions from old. For example, as far as we are aware, the $D=4,\mathcal{N}_-=1$ twins have not appeared as double-copies previously. In particular, for $\mathcal{N}=1$ and $\tilde{\mathcal{N}}=1$ we obtain $\mathcal{N}_+=2$ supergravity minimally coupled to a single hypermultiplet with scalar coset $\mathrm{U}(1,2)/\mathrm{U}(2)$, which was double-copy constructed in [18], but its twin, $\mathcal{N}_-=1$ supergravity minimally coupled to a single vector multiplet and two chiral multiplets, has not yet appeared and remains to be tested at loop level. There is in fact a two parameter family of $(2,1)$ twins coupled to vector and hyper multiplets [4], which do not belong to the pyramid, but can be double-copy

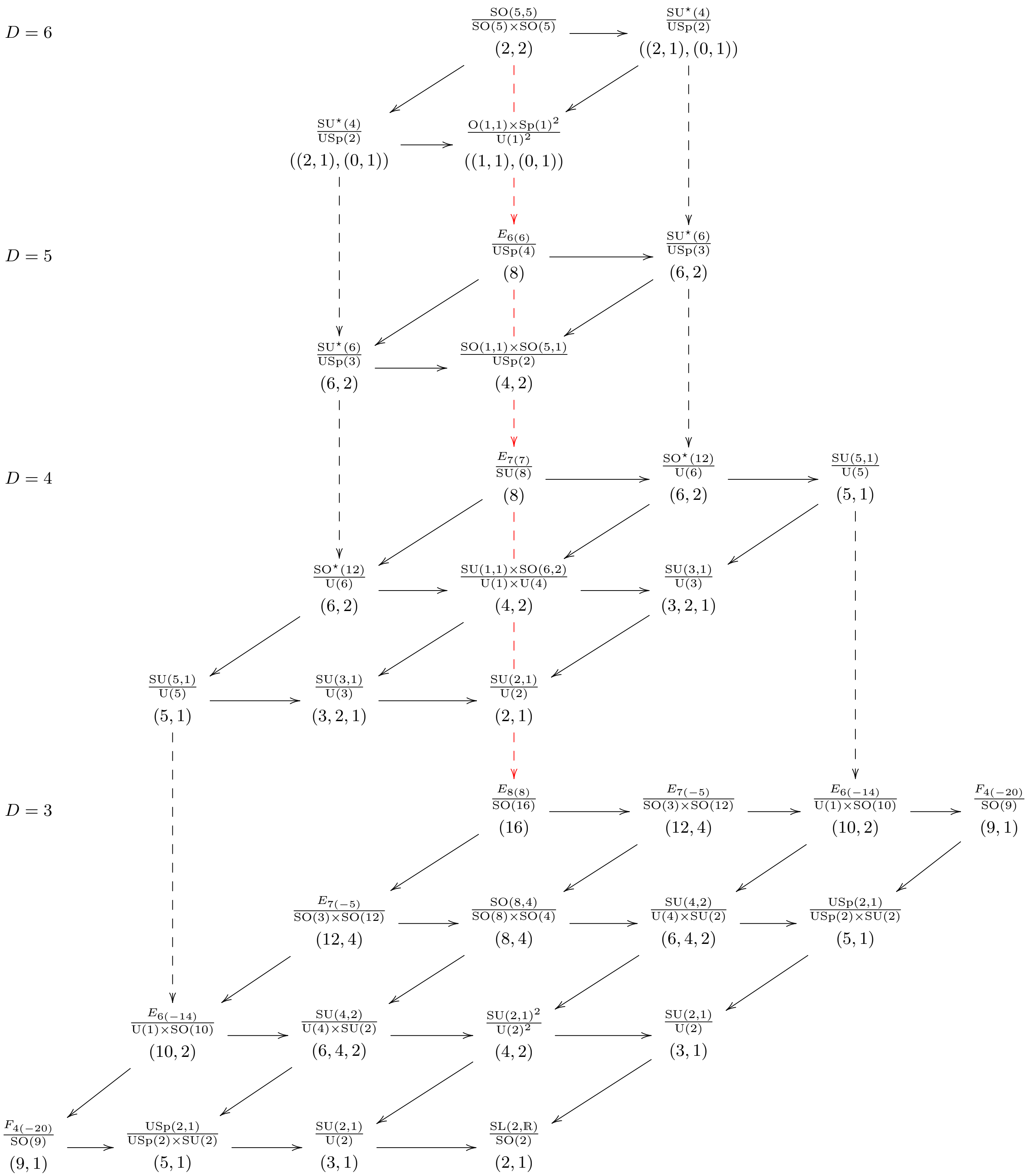

TABLE IV. Pyramid of twin supergravities generated by the product of left and right super Yang-Mills theories in $D = 3, 4, 5, 6$. Each level is related by dimensional reduction as indicated by the vertical arrows. The horizontal arrows indicate consistent truncations effected by truncating the left or right Yang-Mills multiplets. The twins and triplets are indicated by $(\mathcal{N}_+, \mathcal{N}_-)$ and $(\mathcal{N}_+, \mathcal{N}_-^+, \mathcal{N}_-^-)$, respectively, together with their common scalar manifolds. All such supergravity theories have a twin related by their left/right factors except for the maximal cases along the "exceptional spine" highlighted in red. Consequently, for $D > 6$ there are no twin theories and this portion of the pyramid is omitted. Note, $D = 3$ is the exception to the exceptions in that maximal $\mathcal{N} = 16$ supergravity does have a 'trivial' $\mathcal{N} = 1$ twin, but it is not obtained from our double-copy procedure and so is excluded.

TABLE V. The twin supergravities in $D = 3$. Here we give the left and right (super) Yang-Mills products yielding the twin $(\mathcal{N}_+, \mathcal{N}_-)$ supergravities.

Left Yang-Mills-matter			Right Yang-Mills-matter			Twin supergravities			
$\mathcal{N}'$	Content	Symmetry	$\tilde{\mathcal{N}}^{(\prime)}$	Content	Symmetry	$\mathcal{N}_\pm$	Content	Symmetry	Coset
4	$\mathbf{V}_4 \oplus \mathbf{C}_4^\rho$	$\mathfrak{so}(4)_R \oplus \mathfrak{so}(3)_f$	8 0	$\mathbf{V}_8$ $\tilde{A}(\mathbf{1}) \oplus \tilde{\chi}^{\tilde{\rho}}(\mathbf{8}) \oplus \tilde{\phi}(\mathbf{7})$	$\mathfrak{so}(7)$	12 4	$\mathbf{G}_{12}$ $\mathbf{G}_4 \oplus 16\mathbf{V}_4$	$\mathfrak{so}(12)_R \oplus \mathfrak{so}(3)$ $\mathfrak{so}(4)_R \oplus \mathfrak{so}(12)_{\text{Isotropy}} \oplus \mathfrak{so}(3)$	$\frac{E_{7(-5)}}{\mathrm{SO}(12)\times\mathrm{SO}(3)}$
4	$\mathbf{V}_4 \oplus \mathbf{C}_4^\rho$	$\mathfrak{so}(4)_R \oplus \mathfrak{so}(3)_f$	4 0	$\mathbf{V}_4$ $\tilde{A}(\mathbf{1},\mathbf{1}) \oplus \tilde{\chi}^{\tilde{\rho}}(\mathbf{2},\mathbf{2}) \oplus \tilde{\phi}(\mathbf{3},\mathbf{1})$	$\mathfrak{so}(4)$	8 4	$\mathbf{G}_8 \oplus 4\mathbf{V}_8$ $\mathbf{G}_4 \oplus 8\mathbf{V}_4$	$\mathfrak{so}(8)_R \oplus \mathfrak{so}(4)_{\text{Isotropy}}$ $\mathfrak{so}(4)_R \oplus \mathfrak{so}(8)_{\text{Isotropy}} \oplus \mathfrak{so}(4)$	$\frac{\mathrm{SO}(8,4)}{\mathrm{SO}(8)\times\mathrm{SO}(4)}$
2	$\mathbf{V}_2 \oplus \mathbf{C}_2^\rho$	$\mathfrak{so}(2)_r \oplus \mathfrak{so}(2)_f$	8 0	$\mathbf{V}_8$ $\tilde{A}(\mathbf{1}) \oplus \tilde{\chi}^{\tilde{\rho}}(\mathbf{8}) \oplus \tilde{\phi}(\mathbf{7})$	$\mathfrak{so}(7)$	10 2	$\mathbf{G}_{10}$ $\mathbf{G}_2 \oplus \mathbf{V}_2 \oplus 10\mathbf{V}_2 \oplus 5\mathbf{C}_2$	$\mathfrak{so}(10)_R \oplus \mathfrak{so}(2)$ $\mathfrak{so}(2)_R \oplus \mathfrak{so}(10)_{\text{Isotropy}} \oplus \mathfrak{so}(2)$	$\frac{E_{6(-14)}}{\mathrm{SO}(10)\times\mathrm{U}(1)}$
2	$\mathbf{V}_2 \oplus \mathbf{C}_2^\rho$	$\mathfrak{so}(2)_r \oplus \mathfrak{so}(2)_f$	4 0	$\mathbf{V}_4$ $\tilde{A}(\mathbf{1},\mathbf{1}) \oplus \tilde{\chi}^{\tilde{\rho}}(\mathbf{2},\mathbf{2}) \oplus \tilde{\phi}(\mathbf{3},\mathbf{1})$	$\mathfrak{so}(4)$	6 2	$\mathbf{G}_6 \oplus 2\mathbf{V}_6$ $\mathbf{G}_2 \oplus \mathbf{V}_2 \oplus 4\mathbf{V}_2 \oplus 3\mathbf{C}_2$	$\mathfrak{so}(6)_R \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(2)_{\text{Isotropy}}$ $\mathfrak{so}(2)_R \oplus [\mathfrak{su}(4) \oplus \mathfrak{u}(2)]_{\text{Isotropy}}$	$\frac{\mathrm{SU}(4,2)}{\mathrm{SU}(4)\times\mathrm{U}(2)}$
2	$\mathbf{V}_2 \oplus \mathbf{C}_2^\rho$	$\mathfrak{so}(2)_r \oplus \mathfrak{so}(2)_f$	2 0	$\mathbf{V}_2$ $\tilde{A} \oplus 2\tilde{\chi}^{\tilde{\rho}} \oplus \phi$	$\mathfrak{so}(2)$	4 2	$\mathbf{G}_4 \oplus \mathbf{V}_4 \oplus \mathbf{C}_4$ $\mathbf{G}_2 \oplus \mathbf{V}_2 \oplus \mathbf{V}_2 \oplus 2\mathbf{C}_2$	$\mathfrak{so}(4)_R \oplus \mathfrak{so}(2) \oplus \mathfrak{so}(2)_{\text{Isotropy}}$ $\mathfrak{so}(2)_R \oplus [\mathfrak{u}(2) \oplus \mathfrak{u}(2)]_{\text{Isotropy}}$	$\frac{\mathrm{SU}(2,1)}{\mathrm{U}(2)} \times \frac{\mathrm{SU}(2,1)}{\mathrm{U}(2)}$
1	$\mathbf{V}_1 \oplus \mathbf{C}_1^\rho$	$\varnothing$	8 0	$\mathbf{V}_8$ $\tilde{A}(\mathbf{1}) \oplus \tilde{\chi}^{\tilde{\rho}}(\mathbf{8}) \oplus \tilde{\phi}(\mathbf{7})$	$\mathfrak{so}(7)$	9 1	$\mathbf{G}_9$ $\mathbf{G}_1 \oplus 16\mathbf{V}_1$	$\mathfrak{so}(9)_R$ $\mathfrak{so}(9)_{\text{Isotropy}}$	$\frac{F_{4(-20)}}{\mathrm{SO}(9)}$
1	$\mathbf{V}_1 \oplus \mathbf{C}_1^\rho$	$\varnothing$	4 0	$\mathbf{V}_4$ $\tilde{A}(\mathbf{1},\mathbf{1}) \oplus \tilde{\chi}^{\tilde{\rho}}(\mathbf{2},\mathbf{2}) \oplus \tilde{\phi}(\mathbf{3},\mathbf{1})$	$\mathfrak{so}(4)$	5 1	$\mathbf{G}_5 \oplus \mathbf{V}_5$ $\mathbf{G}_1 \oplus 8\mathbf{V}_1$	$\mathfrak{so}(5)_R \oplus \mathfrak{so}(3)$ $\mathfrak{so}(5)_{\text{Isotropy}} \oplus \mathfrak{so}(3)$	$\frac{\mathrm{USp}(2,1)}{\mathrm{USp}(2)\times\mathrm{SU}(2)}$
1	$\mathbf{V}_1 \oplus \mathbf{C}_1^\rho$	$\varnothing$	2 0	$\mathbf{V}_2$ $\tilde{A} \oplus 2\tilde{\chi}^{\tilde{\rho}} \oplus \phi$	$\mathfrak{so}(2)$	3 1	$\mathbf{G}_3 \oplus \mathbf{V}_3$ $\mathbf{G}_1 \oplus 4\mathbf{V}_1$	$\mathfrak{so}(3)_R \oplus \mathfrak{so}(2)$ $\mathfrak{so}(2) \oplus \mathfrak{so}(3)_{\text{Isotropy}}$	$\frac{\mathrm{SU}(2,1)}{\mathrm{U}(2)}$
1	$\mathbf{V}_1 \oplus \mathbf{C}_1^\rho$	$\varnothing$	1 0	$\mathbf{V}_1$ $\tilde{A} \oplus \tilde{\chi}^{\tilde{\rho}}$	$\varnothing$	2 1	$\mathbf{G}_2 \oplus \mathbf{V}_2$ $\mathbf{G}_1 \oplus 2\mathbf{V}_1$	$\mathfrak{so}(2)_R$ $\mathfrak{so}(2)_{\text{Isotropy}}$	$\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}$

TABLE VI. The twin supergravities in $D = 4$. Here we give the left and right (super) Yang-Mills products yielding the twin $(\mathcal{N}_+, \mathcal{N}_-)$ supergravities. In the first column we summarise the left Yang-Mills-matter theories and their global symmetries, where $\mathbf{V}_\mathcal{N}$ and $\mathbf{C}_\mathcal{N}$ are in the adjoint and ρ representations of the left gauge group G respectively. In the second column we summarise the right Yang-Mills-matter theories before and after (10) has been applied. Again, for both cases their global symmetries are given and for the $\tilde{\mathcal{N}} = 0$ theories we have indicated the representation carried by each field (omitting all $\mathfrak{u}(1)$ charges). Note, the fermions of the $\tilde{\mathcal{N}} = 0$ theories are always in the $\tilde{\rho}$ representation of the right gauge group $\tilde{G}$, while the vectors and scalars remain in the adjoint. In the final column we have tabulated the resulting pairs of twin supergravity theories and their common scalar coset manifolds. Note, the final row can be generalised to an infinite sequence of $\mathcal{N}_+ = 2$ self-mirror minimally coupled supergravity theories and their $\mathcal{N}_- = 1$ twins, as discussed in section III D.

Left Yang-Mills-matter			Right Yang-Mills-matter			Twin supergravities			
$\mathcal{N}'$	Content	Symmetry	$\tilde{\mathcal{N}}^{(\prime)}$	Content	Symmetry	$\mathcal{N}_\pm$	Content	Symmetry	Coset
2	$\mathbf{V}_2 \oplus \mathbf{H}_2^\rho$	$\mathfrak{u}(2)_R \oplus \mathfrak{su}(2)_f$	4 0	$\mathbf{V}_4$ $\tilde{A}(\mathbf{1}) \oplus \tilde{\chi}^{\tilde{\rho}}(\mathbf{4}) \oplus \tilde{\phi}(\mathbf{6})$	$\mathfrak{su}(4)_R$ $\mathfrak{su}(4)$	6 2	$\mathbf{G}_6$ $\mathbf{G}_2 \oplus \mathbf{V}_2(\mathbf{15})$	$\mathfrak{u}(6)_R$ $\mathfrak{u}(2)_R \oplus \mathfrak{u}(6)_{\text{Isotropy}}$	$\frac{\mathrm{SO}^\star(12)}{\mathrm{U}(6)}$
2	$\mathbf{V}_2 \oplus \mathbf{H}_2^\rho$	$\mathfrak{u}(2)_R \oplus \mathfrak{su}(2)_f$	2 0	$\mathbf{V}_2$ $\tilde{A}(\mathbf{1}) \oplus \tilde{\chi}^{\tilde{\rho}}(\mathbf{2}) \oplus 2\tilde{\phi}(\mathbf{1})$	$\mathfrak{u}(2)_R$ $\mathfrak{u}(2)$	4 2	$\mathbf{G}_4 \oplus 2\mathbf{V}_4$ $\mathbf{G}_2 \oplus \mathbf{V}_2(\mathbf{1}) \oplus \mathbf{V}_2(\mathbf{6})$	$\mathfrak{u}(4)_R \oplus \mathfrak{so}(2)_{\text{Isotropy}}$ $\mathfrak{u}(2)_R \oplus \mathfrak{u}(4)_{\text{Isotropy}}$	$\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(6,2)}{\mathrm{SO}(2)\times\mathrm{U}(4)}$
1	$\mathbf{V}_1 \oplus \mathbf{C}_1^\rho$	$\mathfrak{u}(1)_R \oplus \mathfrak{u}(1)_f$	4 0	$\mathbf{V}_4$ $\tilde{A}(\mathbf{1}) \oplus \tilde{\chi}^{\tilde{\rho}}(\mathbf{4}) \oplus \tilde{\phi}(\mathbf{6})$	$\mathfrak{su}(4)_R$ $\mathfrak{su}(4)$	5 1	$\mathbf{G}_5$ $\mathbf{G}_1 \oplus \mathbf{V}_1(\mathbf{10}) \oplus \mathbf{C}_1(\mathbf{5})$	$\mathfrak{u}(5)_R$ $\mathfrak{u}(1)_R \oplus \mathfrak{u}(5)_{\text{Isotropy}}$	$\frac{\mathrm{SU}(5,1)}{\mathrm{U}(5)}$
1	$\mathbf{V}_1 \oplus \mathbf{C}_1^\rho$	$\mathfrak{u}(1)_R \oplus \mathfrak{u}(1)_f$	2 1	$\mathbf{V}_2$ $\tilde{A}(\mathbf{1}) \oplus \tilde{\chi}^{\tilde{\rho}}(\mathbf{2}) \oplus 2\tilde{\phi}(\mathbf{1})$	$\mathfrak{u}(2)_R$ $\mathfrak{u}(2)$	3 1	$\mathbf{G}_3 \oplus \mathbf{V}_3$ $\mathbf{G}_1 \oplus \mathbf{V}_1(\mathbf{1}) \oplus \mathbf{V}_1(\mathbf{3}) \oplus \mathbf{C}_1(\mathbf{3})$	$\mathfrak{u}(3)_R \oplus \mathfrak{u}(1)$ $\mathfrak{u}(1)_R \oplus \mathfrak{u}(3)_{\text{Isotropy}}$	$\frac{\mathrm{U}(3,1)}{\mathrm{U}(3)\times\mathrm{U}(1)}$
1	$\mathbf{V}_1 \oplus \mathbf{C}_1^\rho$	$\mathfrak{u}(1)_R \oplus \mathfrak{u}(1)_f$	1 0	$\mathbf{V}_1$ $\tilde{A} \oplus \tilde{\chi}^{\tilde{\rho}}$	$\mathfrak{u}(1)_R$ $\mathfrak{u}(1)$	2 1	$\mathbf{G}_2 \oplus \mathbf{H}_2$ $\mathbf{G}_1 \oplus \mathbf{V}_1(\mathbf{1}) \oplus \mathbf{C}_1(\mathbf{2})$	$\mathfrak{u}(2)_R \oplus \mathfrak{u}(1)$ $\mathfrak{u}(1)_R \oplus \mathfrak{u}(2)_{\text{Isotropy}}$	$\frac{\mathrm{U}(2,1)}{\mathrm{U}(2)\times\mathrm{U}(1)}$

TABLE VII. The twin supergravities in $D = 5$. Here we give the left and right (super) Yang-Mills products yielding the twin $(\mathcal{N}_+, \mathcal{N}_-)$ supergravities.

Left Yang-Mills-matter			Right Yang-Mills-matter			Twin supergravities			
$\mathcal{N}'$	Content	Symmetry	$\tilde{\mathcal{N}}^{(\prime)}$	Content	Symmetry	$\mathcal{N}_\pm$	Content	Symmetry	Coset
2	$\mathbf{V}_2 \oplus \mathbf{H}_2^\rho$	$\mathfrak{sp}(1)_R \oplus \mathfrak{sp}(1)_f$	4 0	$\tilde{\mathbf{V}}_4$ $\tilde{A}(\mathbf{1}) \oplus \tilde{\chi}^{\tilde{\rho}}(\mathbf{4}) \oplus \tilde{\phi}(\mathbf{5})$	$\mathfrak{sp}(2)_R$ $\mathfrak{sp}(2)$	6 2	$\mathbf{G}_6$ $\mathbf{G}_2 \oplus \mathbf{V}_2(\mathbf{14})$	$\mathfrak{sp}(3)_R$ $\mathfrak{sp}(1)_R \oplus \mathfrak{sp}(3)_{\text{Isotropy}}$	$\frac{\text{SU}^*(6)}{\text{USp}(3)}$
2	$\mathbf{V}_2 \oplus \mathbf{H}_2^\rho$	$\mathfrak{sp}(1)_R \oplus \mathfrak{sp}(1)_f$	2 0	$\tilde{\mathbf{V}}_2$ $\tilde{A}(\mathbf{1}) \oplus \tilde{\chi}^{\tilde{\rho}}(\mathbf{2}) \oplus \tilde{\phi}(\mathbf{1})$	$\mathfrak{sp}(1)_R$ $\mathfrak{sp}(1)$	4 2	$\mathbf{G}_4 \oplus \mathbf{V}_4$ $\mathbf{G}_2 \oplus \mathbf{V}_2(\mathbf{1}) \oplus \mathbf{V}_2(\mathbf{5})$	$\mathfrak{sp}(2)_R$ $\mathfrak{sp}(1)_R \oplus \mathfrak{sp}(2)_{\text{Isotropy}}$	$\frac{\text{O}(5,1)\times\text{O}(1,1)}{\text{USp}(2)}$

TABLE VIII. The twin supergravities in $D = 6$. The $\mathcal{N}_+$ twin is generated as a truncation of the parent as for $D = 3, 4, 5$. The $\mathcal{N}_-$ twin requires an additional chirality flip of the left Yang-Mills-matter multiplet.

Left Yang-Mills-matter			Right Yang-Mills-matter			Twin supergravities			
$\mathcal{N}'$	Content	Symmetry	$\tilde{\mathcal{N}}^{(\prime)}$	Content	Symmetry	$\mathcal{N}_\pm$	Content	Symmetry	Coset
$(1,0)$ $(0,1)$	$\mathbf{V}_{1,0} \oplus \mathbf{H}^\rho_{1,0}$ $\mathbf{V}_{0,1} \oplus \mathbf{H}^\rho_{0,1}$	$\mathfrak{sp}(1)_R \oplus \mathfrak{sp}(1)_f$ $\mathfrak{sp}(1)_f \oplus \mathfrak{sp}(1)_R$	$(1,1)$ $(0,0)$	$\tilde{\mathbf{V}}_{1,1}$ $\tilde{A}(\mathbf{1}) \oplus \tilde{\chi}^\rho_-(\mathbf{2}) \oplus \tilde{\chi}^\rho_+(\mathbf{2}) \oplus \tilde{\phi}(\mathbf{4})$	$\mathfrak{sp}(1)_R \oplus \mathfrak{sp}(1)_R$ $\mathfrak{sp}(1)_R$	$(2,1)$ $(0,1)$	$\mathbf{G}_{2,1}$ $\mathbf{G}_{0,1} \oplus \mathbf{V}_{0,1}(\mathbf{4}+\mathbf{4}) \oplus \mathbf{T}_{0,1}(\mathbf{5})$	$\mathfrak{sp}(2)_R \oplus \mathfrak{sp}(1)_R$ $\mathfrak{sp}(1)_R \oplus \mathfrak{sp}(2)_{\text{Isotropy}}$	$\frac{\text{SU}^*(4)\times\text{Sp}(1)}{\text{USp}(2)\times\text{U}(1)}$
$(1,0)$ $(0,1)$	$\mathbf{V}_{1,0} \oplus \mathbf{H}^\rho_{1,0}$ $\mathbf{V}_{0,1} \oplus \mathbf{H}^\rho_{0,1}$	$\mathfrak{sp}(1)_R \oplus \mathfrak{sp}(1)_f$ $\mathfrak{sp}(1)_f \oplus \mathfrak{sp}(1)_R$	$(1,0)$ $(0,0)$	$\tilde{\mathbf{V}}_{1,0}$ $\tilde{A}(\mathbf{1}) \oplus \tilde{\chi}^\rho_-(\mathbf{2})$	$\mathfrak{sp}(1)_R$ $\mathfrak{sp}(1)$	$(2,0)$ $(0,1)$	$\mathbf{G}_{2,0} \oplus \mathbf{T}_{2,0}$ $\mathbf{G}_{0,1} \oplus \mathbf{T}_{0,1}(\mathbf{5})$	$\mathfrak{sp}(2)_R$ $\mathfrak{sp}(1)_R \oplus \mathfrak{sp}(2)_{\text{Isotropy}}$	$\frac{\text{SU}^*(4)}{\text{USp}(2)}$
$(1,0)$ $(0,1)$	$\mathbf{V}_{1,0} \oplus \mathbf{H}^\rho_{1,0}$ $\mathbf{V}_{0,1} \oplus \mathbf{H}^\rho_{0,1}$	$\mathfrak{sp}(1)_R \oplus \mathfrak{sp}(1)_f$ $\mathfrak{sp}(1)_f \oplus \mathfrak{sp}(1)_R$	$(0,1)$ $(0,0)$	$\tilde{\mathbf{V}}_{0,1}$ $\tilde{A}(\mathbf{1}) \oplus \tilde{\chi}^{\tilde{\rho}}_+(\mathbf{2})$	$\mathfrak{sp}(1)_R$ $\mathfrak{sp}(1)$	$(1,1)$ $(0,1)$	$\mathbf{G}_{1,1}$ $\mathbf{G}_{0,1} \oplus \mathbf{V}_{0,1}(\mathbf{4}) \oplus \mathbf{T}_{0,1}(\mathbf{1})$	$\mathfrak{sp}(1)_R \oplus \mathfrak{sp}(1)_R$ $\mathfrak{sp}(1)_R \oplus \mathfrak{sp}(1)_{\text{Isotropy}}$	$\frac{\text{O}(1,1)\times\text{Sp}(1)^2}{\text{U}(1)^2}$

constructed as discussed in section III D. Note, the associated sequence of special Kähler symmetric scalar manifolds appearing in the $\mathcal{N}_+ = 2$ theories can also be coupled to $\mathcal{N}_- = 1, D = 4$ supergravity because their kinetic vector matrices are holomorphic [77].

Second, our approach applied to the prototypical $D = 4, (6,2)$ twin pair gives an alternative double-copy construction of the quaternionic magic $D = 4, \mathcal{N} = 2$ supergravity, which was previously obtained in [18] using a different pair of Yang-Mills-matter factors. This serves to highlight a general feature of the double-copy construction for matter-coupled supergravities: the factorisation into left and right (super) Yang-Mills multiplets is not necessarily unique. The $D = 4, (4,2)$ twin pair is a clear example. The $\mathcal{N}_+ = 4$ supergravity comes coupled to two vector multiplets and follows from the product of two, $\mathcal{N} = \tilde{\mathcal{N}} = 2$, super Yang-Mills multiplets. As a truncation of the parent $\mathcal{N} = 6$ theory it is schematically given by,

$$[\mathbf{V}_4] \otimes [\tilde{\mathbf{V}}_2] = \mathbf{G}_6 \longrightarrow [\mathbf{V}_2 \oplus \mathbf{C}_2^\rho] \otimes [\tilde{\mathbf{V}}_2] = \mathbf{G}_4 \oplus 2\mathbf{V}_4. \tag{53}$$

Its twin $\mathcal{N}_- = 2$ supergravity is coupled to seven vector multiplets and follows from the same principle applied to $\tilde{\mathbf{V}}_2$,

$$\begin{aligned}[\mathbf{V}_2 \oplus \mathbf{C}_2^\rho] \otimes [\tilde{\mathbf{V}}_2] \longrightarrow [\mathbf{V}_2 \oplus \mathbf{C}_2^\rho] \otimes [\tilde{A} \oplus \tilde{\chi}_a^{\tilde{\rho}} \oplus \tilde{\phi}] &= [\mathbf{V}_2 \otimes \tilde{A}] \oplus [\mathbf{C}_2^\rho \otimes \tilde{\chi}_a^{\tilde{\rho}}] \oplus [\mathbf{V}_2 \oplus \tilde{\phi}] \\ &= [\mathbf{G}_2 \oplus \mathbf{V}_2] \oplus [2\mathbf{V}_{2a}] \oplus [2\mathbf{V}_2],\end{aligned} \tag{54}$$

where $a = 1, 2$ is an $\mathfrak{su}(2)$ index, the remnant $\tilde{\mathcal{N}} = 2$ R-symmetry, and $\tilde{\phi}$ is a complex scalar. The common scalar coset is given by $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(6,2)/\,\mathrm{U}(1) \times \mathrm{U}(4)$. The $\mathrm{U}(4)$ is the R-symmetry of the $\mathcal{N}_+ = 4$ theory and the matter isotropy group of the $\mathcal{N}_- = 2$ theory, rotating the six vector multiplets with a matter$\otimes$matter origin. Both theories, however, admit an alternative construction [68],

$$\mathbf{G}_4 \oplus 2\mathbf{V}_4 = \mathbf{V}_4 \otimes [\tilde{A} \oplus 2\tilde{\phi}] \quad \text{and} \quad \mathbf{G}_2 \oplus 7\mathbf{V}_2 = \mathbf{V}_2 \otimes [\tilde{A} \oplus 6\tilde{\phi}], \tag{55}$$

where the n scalar fields of the right multiplets are required to transform in the vector representation of $\mathrm{SO}(n)$. It turns out there is a plethora of non-unique decompositions of this type. The full classification of all supergravities admitting more than one Yang-Mills factorisations will be given in [78].

The $D = 6$ case, given in Table VIII, is a little more subtle. In particular, we must take to account the possible chiralities, $\mathcal{N} = (n, m)$. The big twin is obtained as a truncation of its parent following the prescription laid out above. To obtain the little twin, however, the decomposition of the right Yang-Mills multiplet must be accompanied by a flip of the chirality of the left Yang-Mills-matter multiplet:

$$\begin{array}{llcll}\text{Parent} & \mathbf{V}_{(n,m)} & \otimes & \tilde{\mathbf{V}}_{(\tilde{n},\tilde{m})} & = \mathbf{G}_{(n+\tilde{n},m+\tilde{m})} \oplus \mathbf{M}_{(n+\tilde{n},m+\tilde{m})} \\ \text{Big twin} & \mathbf{V}_{(n',m')} \oplus \mathbf{H}^\rho_{(n',m')} & \otimes & \tilde{\mathbf{V}}_{(\tilde{n},\tilde{m})} & = \mathbf{G}_{(n_+,m_+)} \oplus \mathbf{M}_{(n_+,m_+)} \\ \text{Little twin} & \underbrace{\mathbf{V}_{(m',n')} \oplus \mathbf{H}^\rho_{(m',n')}}_{\text{Chirality flipped}} & \otimes & \tilde{\mathbf{V}}_{(\tilde{n}',\tilde{m}')} \oplus \tilde{\mathbf{C}}^\rho_{(\tilde{n}',\tilde{m}')} \oplus \cdots & = \mathbf{G}_{(n_-,m_-)} \oplus \mathbf{M}_{(n_-,m_-)}\end{array} \tag{56}$$

For example, the $(\mathcal{N}_+, \mathcal{N}_-) = ((2,1),(0,1))$ is given by

$$\begin{array}{llcll}\text{Parent} & \mathbf{V}_{(1,1)} & \otimes & \tilde{\mathbf{V}}_{(1,1)} & = \mathbf{G}_{(2,2)} \\ \text{Big twin} & \mathbf{V}_{(1,0)} \oplus \mathbf{H}^\rho_{(1,0)} & \otimes & \tilde{\mathbf{V}}_{(1,1)} & = \mathbf{G}_{(2,1)} \\ \text{Little twin} & \mathbf{V}_{(0,1)} \oplus \mathbf{H}^\rho_{(0,1)} & \otimes & \tilde{A} \oplus 2(\tilde{\chi}^\rho_-, \tilde{\chi}^\rho_+) \oplus 4\tilde{\phi} & = \mathbf{G}_{(0,1)} \oplus 8\mathbf{V}_{(0,1)} \oplus 5\mathbf{V}_{(0,1)}\end{array} \tag{57}$$

All three cases are presented in Table VIII.

Note, the $(\mathcal{N}_+, \mathcal{N}_-) = ((2,1),(0,1))$ example can also be generated by using tensor multiplets $\mathbf{T}_{(n,m)}$, at least at the level of free on-shell states:

$$\begin{array}{llcll}\text{Parent} & \mathbf{T}_{(0,2)} & \otimes & \tilde{\mathbf{T}}_{(2,0)} & = \mathbf{G}_{(2,2)} \\ \text{Big twin} & \mathbf{T}_{(0,1)} \oplus \mathbf{H}^\rho_{(0,1)} & \otimes & \tilde{\mathbf{T}}_{(2,0)} & = \mathbf{G}_{(2,1)} \\ \text{Little twin} & \mathbf{T}_{(0,1)} \oplus \mathbf{H}^\rho_{(0,1)} & \otimes & \tilde{B} \oplus 4\tilde{\chi}^\rho_+ \oplus 5\tilde{\phi} & = \mathbf{G}_{(0,1)} \oplus 8\mathbf{V}_{(0,1)} \oplus 5\mathbf{T}_{(0,1)}\end{array} \tag{58}$$

The $D = 6$, $\mathbf{G}_{(2,2)}$ multiplet is the unique maximally supersymmetric gravity multiplet that admits two factorisations,

$$\mathbf{G}_{(2,2)} = \mathbf{T}_{(2,0)} \otimes \tilde{\mathbf{T}}_{(0,2)}, \qquad \mathbf{G}_{(2,2)} = \mathbf{V}_{(1,1)} \otimes \tilde{\mathbf{V}}_{(1,1)}. \tag{59}$$

Although the $D=6,(2,0)$ theories are intrinsically 'strongly coupled' and do not admit any conventional Lagrangian description, the existence of well-defined asymptotic states facilitates a direct analysis of the S-matrix [96]. The tree-level amplitudes may be defined as the purely pole part of the S-matrix, although strong coupling implies that they cannot be interpreted as the leading term in any perturbative expansion. In the conformal phase there are no non-vanishing tree-level amplitudes for the self-dual tensor that respect the $(2,0)$ super-Poincaré invariance [96, 97], leaving the double-copy origin of the $\mathbf{G}_{(2,2)}$ amplitudes mysterious from this perspective. One approach is to consider M5-branes on $\mathbb{R}^{1,4}\times S^1$ with self-dual strings (M2-branes ending on the M5-branes) wrapping the S^1 to give a tower of massive Kaluza-Klein modes in five dimensions. Then there is a non-trivial three-point amplitude for the self-dual tensor, which squares to give an amplitude of the $D=6,(4,0)$ theory on $\mathbb{R}^{1,4}\times S^1$ [97]. In the massless limit the self-dual tensor amplitude reduces to that of $D=5,\mathcal{N}=4$ super Yang-Mills so that its square correctly produces the corresponding $D=5,\mathcal{N}=8$ supergravity amplitude [97]. This is consistent with the observation that in the linear approximation the $(4,0)$ theory dimensionally reduced on a circle is $D=5,\mathcal{N}=8$ supergravity [98, 99]. Alternatively, the product of the $(2,0)$ and $(0,2)$ amplitudes gives that of $D=6,\mathcal{N}=8$ supergravity on $\mathbb{R}^{1,4}\times S^1$, as suggested by (59), which also gives the $D=5,\mathcal{N}=8$ supergravity amplitude in the massless limit.

Finally, a comment on $D=3,\mathcal{N}=1$ theories is needed. As noted in [3] and section II all $\mathcal{N}_+>1$ theories (assuming all vectors have been dualised to scalars) in $D=3$ have an $\mathcal{N}_-=1$ twin. This follows from the fact that $D=3,\mathcal{N}=1$ supergravity can be coupled to scalars parametrising any Riemannian manifold and all admissible scalar cosets for $\mathcal{N}>1$ are Riemannian. For this reason such twins are typically regarded as trivial. However, for the pyramid of twins, Table IV, they are natural in the sense that they follow from the same double-copy construction described. Note however, the $D=3,(16,1)$ twin pair is *not* obtained in this manner and, as such, it should be regarded as belonging to the excluded maximal spine. It is also excluded on the basis that the $D=3,(16,1)$ twins do not have a parent supergravity.

C. The triplets

In four dimensions there is a $(\mathcal{N}_+,\mathcal{N}_-^+,\mathcal{N}_-^-)=(3,2,1)$ triplet of supergravity theories, which descends to a $(6,4,2)$ triplet in $D=3$. The notation $\mathcal{N}_-^\pm$ is used to indicate that $\mathcal{N}_+$ is the big sibling of both $\mathcal{N}_-^\pm$, while $\mathcal{N}_-^-$ is the little sibling of $\mathcal{N}_-^+$. The common scalar manifold is

$$\frac{\mathrm{SU}(3,1)}{\mathrm{SU}(3)\times\mathrm{U}(1)}. \tag{60}$$

We find that the two sub-twins $(\mathcal{N}_+,\mathcal{N}_-^+)$ and $(\mathcal{N}_+,\mathcal{N}_-^-)$ follow from the same considerations as above, as the $\mathcal{N}_+=3$ theory belongs to the pyramid. Specifically, for the $(3,2)$ pair we have an $\mathcal{N}=5$ parent,

$$\begin{array}{llcccl}
\text{Parent} & \mathbf{V}_4 & \otimes & \tilde{\mathbf{V}}_1 & = & \mathbf{G}_5\\
\text{Big twin} & \mathbf{V}_2\oplus\mathbf{H}_2^\rho & \otimes & \tilde{\mathbf{V}}_1 & = & \mathbf{G}_3\oplus\mathbf{V}_3\\
\text{Little twin} & \mathbf{V}_2\oplus\mathbf{H}_2^\rho & \otimes & \tilde{A}\oplus\tilde{\chi}^\rho & = & \mathbf{G}_2\oplus 3\mathbf{V}_2
\end{array} \tag{61}$$

The left and right symmetries of the big twin factors are

$$\mathfrak{so}(2)_l\oplus\mathfrak{u}(2)_R\oplus\mathfrak{su}(2)_f \quad\text{and}\quad \mathfrak{so}(2)_r\oplus\mathfrak{u}(1)_R, \tag{62}$$

where $\mathfrak{u}(2)_R$ is the left $\mathcal{N}'=2$ R-symmetry and $\mathfrak{su}(2)_f$ is a remnant of the $\mathcal{N}=4$ R-symmetry feeding into the $\mathcal{N}=5$ parent. They sit inside the $\mathcal{N}=3$ algebra as

$$[\mathfrak{u}(2)_R\oplus\mathfrak{u}(1)_R]\oplus\mathfrak{u}(1)\subset\mathfrak{u}(3)_R\oplus\mathfrak{u}(1)_{\text{Isotropy}}, \tag{63}$$

where the additional $\mathfrak{u}(1)$ is given by the difference of the $\mathfrak{so}(2)_l$ and $\mathfrak{so}(2)_r$ generators as usual. Note, the $\mathfrak{su}(2)_f$ acts trivially on all gravitational states as it only acts non-trivially on the $\mathbf{H}_2^\rho$ multiplet, which plays no role here.

Similarly, the left and right symmetries of the little twin factors are

$$\mathfrak{so}(2)_l\oplus\mathfrak{u}(2)_R\oplus\mathfrak{su}(2)_f \quad\text{and}\quad \mathfrak{so}(2)_r\oplus\mathfrak{u}(1)_f, \tag{64}$$

where the right $\mathfrak{u}(1)_f$ is now a remnant of the right $\tilde{\mathcal{N}}=1$ R-symmetry. They sit inside the $\mathcal{N}=2$ algebra as

$$\mathfrak{u}(2)_R\oplus[\mathfrak{su}(2)_f\oplus\mathfrak{u}(1)_f]\oplus\mathfrak{u}(1)\subset\mathfrak{u}(2)_R\oplus[\mathfrak{su}(3)\oplus\mathfrak{u}(1)]_{\text{Isotropy}} \tag{65}$$

where again the additional $\mathfrak{u}(1)$ is given by the difference of the $\mathfrak{so}(2)_l$ and $\mathfrak{so}(2)_r$ generators. In this case both the $\mathfrak{su}(2)_f$ and $\mathfrak{u}(1)_f$ act non-trivially on the gravitational states in the $\mathbf{H}_2^\rho \otimes \tilde{\chi}^\rho$ sector, while the left R-symmetry $\mathfrak{u}(2)_R$ goes along for the ride to become the gravitational R-symmetry.

For the $(3,1)$ pair we have an $\mathcal{N}=4$ parent,

$$\begin{array}{llcll} \text{Parent} & \mathbf{V}_2 & \otimes & \tilde{\mathbf{V}}_2 & = \mathbf{G}_4 \oplus 2\mathbf{V}_4 \\ \text{Big twin} & \mathbf{V}_1 \oplus \mathbf{C}_1^\rho & \otimes & \tilde{\mathbf{V}}_2 & = \mathbf{G}_3 \oplus \mathbf{V}_3 \\ \text{Little twin} & \mathbf{V}_1 \oplus \mathbf{C}_1^\rho & \otimes & \tilde{A} \oplus 2\tilde{\chi}^\rho \oplus 2\phi & = \mathbf{G}_1 \oplus 4\mathbf{V}_1 \oplus 3\mathbf{C}_1 \end{array} \tag{66}$$

In this case, the left and right symmetries of the big twin factors are

$$\mathfrak{so}(2)_l \oplus \mathfrak{u}(1)_R \oplus \mathfrak{u}(1)_f \quad \text{and} \quad \mathfrak{so}(2)_r \oplus \mathfrak{u}(2)_R, \tag{67}$$

where the right $\mathfrak{u}(1)_f$ is a remnant of the left $\mathcal{N}=2$ R-symmetry. They sit inside the $\mathcal{N}=3$ algebra as

$$[\mathfrak{u}(2)_R \oplus \mathfrak{u}(1)_R] \oplus \mathfrak{u}(1) \subset \mathfrak{u}(3)_R \oplus \mathfrak{u}(1)_{\text{Isotropy}} \tag{68}$$

where as before the additional $\mathfrak{u}(1)$ is given by the difference of the $\mathfrak{so}(2)_l$ and $\mathfrak{so}(2)_r$ generators. Again, all gravitational states are uncharged under the $\mathfrak{u}(1)_f$ as it only acts non-trivially on $\mathbf{C}_1^\rho$.

The left and right symmetries of the little twin factors are

$$\mathfrak{so}(2)_l \oplus \mathfrak{u}(1)_R \oplus \mathfrak{u}(1)_f \quad \text{and} \quad \mathfrak{so}(2)_r \oplus \mathfrak{u}(2)_f, \tag{69}$$

where the right $\mathfrak{u}(2)_f$ is a remnant of the right $\tilde{\mathcal{N}}=2$ R-symmetry. They sit inside the $\mathcal{N}=1$ algebra as

$$\mathfrak{u}(1)_R \oplus [\mathfrak{u}(2)_f \oplus \mathfrak{u}(1)_f] \oplus \mathfrak{u}(1) \subset \mathfrak{u}(1)_R \oplus [\mathfrak{u}(3) \oplus \mathfrak{u}(1)]_{\text{Isotropy}} \tag{70}$$

The extra $\mathfrak{u}(1)$ is given by the difference of the $\mathfrak{so}(2)_l$ and $\mathfrak{so}(2)_r$ generators as in the other cases. In this case both the $\mathfrak{u}(2)_f$ and $\mathfrak{u}(1)_f$ act non-trivially on the gravitational states in the $\mathbf{C}_1^\rho \otimes 2\tilde{\chi}^\rho$ sector, while the left R-symmetry $\mathfrak{u}(1)_R$ becomes the gravitational R-symmetry. Note, there is an additional global $\mathfrak{u}(1)$, which acts trivially on the scalars [84].

Finally, for the $(2,1)$ pair we have an $\mathcal{N}=3$ parent that is not simply the product of pure $\mathcal{N}=2$ and $\mathcal{N}=1$ Yang-Mills,

$$\begin{array}{llcll} \text{Parent} & \mathbf{V}_2 \oplus \mathbf{H}_2^\rho & \otimes & \tilde{\mathbf{V}}_1 \oplus \tilde{\mathbf{C}}_1^{\tilde{\rho}} & = \mathbf{G}_3 \oplus 3\mathbf{V}_3 \\ \text{Big twin} & \mathbf{V}_2 & \otimes & \tilde{A} \oplus 2\tilde{\chi}^{\tilde{\rho}} \oplus 2\tilde{\phi} & = \mathbf{G}_2 \oplus 3\mathbf{V}_2 \\ \text{Little twin} & \mathbf{V}_1 \oplus \mathbf{C}_1^\rho & \otimes & \tilde{A} \oplus 2\tilde{\chi}^\rho \oplus 2\phi & = \mathbf{G}_1 \oplus 4\mathbf{V}_1 \oplus 3\mathbf{C}_1 \end{array} \tag{71}$$

Note, to obtain the $\mathcal{N}_-^+=2$ twin we both decompose the right and truncate the left $\mathbf{H}_2^\rho$ multiplet. Importantly, the $\mathcal{N}=3$ symmetry generated by $\mathbf{V}_2 \otimes \tilde{\mathbf{V}}_1$ alone is not enough to accommodate the required $\mathcal{N}_-^+=2$ R-symmetry plus isotropy, hence the need group theoretically for the two additional $\mathbf{V}_3$. Of course, they are also required for the correct content.

Despite the fact that all three triplets $(3,2,1)$ are truncations of either the $\mathcal{N}=5$ or $\mathcal{N}=4$ parent, symmetry considerations imply that in terms of Yang-Mills-matter factorisations considered here the $(3,1)$ and $(3,2)$ twins can only be accommodated by the $\mathcal{N}=4$ and $\mathcal{N}=5$ parents respectively.

We note that the squaring approach provides a finer graining of the triplet than that of supergravity. From a supergravity perspective, the three sub twin pairs are truly degenerate in the triplet: treating $(3,2,1)$ as a triplet or as three pairs is equivalent. However, from the point of view of the double copy, the triplet degeneracy is partially resolved: not only do the $\mathcal{N}=2$ and $\mathcal{N}=3$ theories admit two different factorisations each, but these are in a $1-1$ correspondence with the sub-twin pair that they belong to. Thus, different factorisations uniquely lead to different sub pairs, and therefore different parents. As an example, given the $\mathcal{N}=2$ theory along with its factorisation in (61), one can determine that it belongs to the sub pair $(3,2)$, while the factorisation in (71) can only sit in the $(2,1)$ sub pair; this distinction cannot be actuated in supergravity. So far, only one factorisation of the $\mathcal{N}=1$ has been found, such that the triplet is not yet fully resolved into three distinct sub pairs. We refer the reader to [78] regarding the possibility of an alternative factorisation of the $\mathcal{N}=1$ theory.

Dimensionally reducing to $D=3$ we obtain a $(6,4,2)$ triplet with common scalar coset (with vectors dualised to scalars),

$$\frac{\text{SU}(4,2)}{\text{U}(4)\times\text{SU}(2)}. \tag{72}$$

D. Other twins

There is an isolated pair of twin supergravities that do not belong to the pyramid since the $\mathcal{N}_+$ twin is not a product of two pure super Yang-Mills theories: the $D = 4, (4, 1)$ pair consisting of pure $\mathcal{N}_+ = 4$ supergravity and $\mathcal{N}_- = 1$ supergravity coupled to $n_V = 6$ vector multiplets and a single chiral multiplet as discussed in section II. Recall, both have U-duality group $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(6)$ and the common scalar manifold is $\mathrm{SU}(1, 1)/\mathrm{U}(1)$. Note however, the pure $\mathcal{N}_+ = 4$ theory *can* be considered as part of the pyramid if the $\mathcal{N} = 0$ vector multiplet is included in the factors,

$$\mathbf{V}_4 \otimes \tilde{A} = \mathbf{G}_4, \qquad \text{with} \qquad [\mathfrak{so}(2)_l \oplus \mathfrak{su}(4)] \oplus [\mathfrak{so}(2)_r] \to \mathfrak{so}(2)_{st} \oplus \mathfrak{u}(4), \tag{73}$$

where the spacetime helicity group $\mathfrak{so}(2)_{st}$ and the additional $\mathfrak{u}(1)$ are given by the sum and difference of the $\mathfrak{so}(2)_l$ and $\mathfrak{so}(2)_r$ generators, respectively. The simplest approach[9] is to take $\mathcal{N} = 5$ supergravity as the parent:

$$\begin{array}{llcll} \text{Parent} & \mathbf{V}_4 & \otimes & \tilde{\mathbf{V}}_1 & = \mathbf{G}_5 \\ \text{Big twin} & \mathbf{V}_4 & \otimes & \tilde{A} \oplus \tilde{\chi}^{\tilde{\rho}} & = \mathbf{G}_4 \\ \text{Little twin} & A \oplus 4\chi^{\rho} \oplus 6\phi & \otimes & \tilde{\mathbf{V}}_1 & = \mathbf{G}_1 \oplus 6\mathbf{V}_1 \oplus \mathbf{C}_1 \end{array} \tag{74}$$

where, departing from the pyramid twins, we decompose either the left or right multiplet, but not both. As usual the R-symmetry of the $\mathcal{N}_+ = 4$ theory becomes the isotropy group of the $\mathcal{N}_- = 1$ theory.

Finally, we recall that the $D = 4, (2, 1)$ twin pair appearing in the pyramid admits a generalisation to a two integer parameter sequence of $(2, 1)$ twins, as pointed out in [4]. The $\mathcal{N}_+ = 2$ sequence is given by $\mathcal{N} = 2$ supergravity minimally coupled to n_V vector multiplets and n_H hyper multiplets. The $\mathcal{N}_- = 1$ sequence is given by $\mathcal{N} = 1$ supergravity coupled to $n_V + 1$ vector multiplets and $n_C = n_V + 2n_H$ chiral multiplets. The common scalar coset is $\mathrm{U}(1, n_V) \times \mathrm{SU}(2, n_H)/[\mathrm{U}(1) \times \mathrm{U}(n_V) \times \mathrm{U}(2) \times \mathrm{SU}(n_H)]$. Arbitrary n_V, n_H requires a sequence of $\mathcal{N} = 3$ parent supergravities coupled to $n = n_V + n_H$ vector multiplets. The $\mathcal{N}_\pm$ scalar cosets then embed into the parent $\mathcal{N} = 3$ scalar coset,

$$\frac{\mathrm{U}(1, n_V) \times \mathrm{U}(2, n_H)}{\mathrm{U}(1) \times \mathrm{U}(n_V) \times \mathrm{U}(2) \times \mathrm{U}(n_H)} \subset \frac{\mathrm{U}(3, n)}{\mathrm{U}(3) \times \mathrm{U}(n)}. \tag{75}$$

With two supersymmetric factors this is achieved by including fundamental matter from the outset,

$$[\mathbf{V}_2 \oplus \mathbf{C}_2^{\rho}] \otimes [\tilde{\mathbf{V}}_1 \oplus m\tilde{\mathbf{C}}_1^{\rho}] = \mathbf{G}_3 \oplus n\mathbf{V}_3, \tag{76}$$

where $n = m + 1$ and

$$[\mathfrak{so}(2)_l \oplus \mathfrak{u}(2)_R] \oplus [\mathfrak{so}(2)_r \oplus \mathfrak{u}(1)_R \oplus \mathfrak{u}(m)] \to \mathfrak{so}(2)_{st} \oplus [\mathfrak{u}(3)_R \oplus \mathfrak{u}(n)]. \tag{77}$$

To obtain the $\mathcal{N}_+ = 2$ twin sequence as a truncation of the $\mathcal{N} = 3$ parent sequence we decompose the right multiplet,

$$\tilde{\mathbf{V}}_1 \oplus m\tilde{\mathbf{C}}_1^{\rho} \longrightarrow \tilde{A} \oplus \tilde{\chi} \oplus m\tilde{\chi}^{\tilde{\rho}} \oplus m\tilde{\phi}^{\tilde{\rho}}, \tag{78}$$

where $\tilde{\phi}$ is complex. We then further truncate $n_H + 1 = n - (n_V - 1)$ of the spinors and $n_V - 1 = n - (n_H + 1)$ of the scalars so that we break to $\mathfrak{u}(n_V - 1) \oplus \mathfrak{u}(n_H) \subset \mathfrak{u}(m)$, leaving

$$[\mathbf{V}_2 \oplus \mathbf{C}_2^{\rho}] \otimes [\tilde{A} \oplus (n_V - 1)\tilde{\chi}^{\tilde{\rho}} \oplus n_H\tilde{\phi}^{\tilde{\rho}}] = \mathbf{G}_2 \oplus n_V\mathbf{V}_2 \oplus n_H\mathbf{H}_2 \tag{79}$$

with symmetries

$$[\mathfrak{so}(2)_l \oplus \mathfrak{u}(2)_R] \oplus [\mathfrak{so}(2)_r \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(n_V - 1) \oplus \mathfrak{u}(n_H)] \to \mathfrak{so}(2)_{st} \oplus [\mathfrak{u}(2)_R \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(n_V) \oplus \mathfrak{u}(n_H)] \tag{80}$$

in agreement with (75).

To obtain the $\mathcal{N}_- = 1$ twin sequence as a truncation of the $\mathcal{N} = 3$ parent sequence we decompose both the left and right multiplets,

$$\begin{aligned} \mathbf{V}_2 \oplus \mathbf{C}_2^{\rho} &\longrightarrow \mathbf{V}_1 \oplus \mathbf{C}_1 \oplus \mathbf{C}_1^{\rho}, \\ \tilde{\mathbf{V}}_1 \oplus m\tilde{\mathbf{C}}_1^{\rho} &\longrightarrow \tilde{A} \oplus \tilde{\chi} \oplus m\tilde{\chi}^{\tilde{\rho}} \oplus m\tilde{\phi}^{\tilde{\rho}}, \end{aligned} \tag{81}$$

[9] This is by no means unique, there is for example an $\mathcal{N} = 4$ parent. We leave the reader to explore the possibilities.

where $\tilde{\phi}$ is complex. Again, we then further truncate both the left and right. On the left we only keep $\tilde{\mathbf{C}}_1^{\rho}$, while on the right we remove $n_H - 1 = n - (n_V + 1)$ of the spinors and $n_V = n - n_H$ of the scalars so that we break to $\mathfrak{u}(1) \oplus \mathfrak{u}(n_V) \oplus \mathfrak{u}(n_H - 1) \subset \mathfrak{u}(m)$, where one of the right $\tilde{\chi}$ is a $\mathfrak{u}(n_V)$ singlet. This yields

$$[\mathbf{V}_1 \oplus \mathbf{C}_1^{\rho}] \otimes [\tilde{A} \oplus \tilde{\chi}^{\tilde{\rho}} \oplus n_V \tilde{\chi}^{\tilde{\rho}} \oplus (n_H - 1)\tilde{\phi}^{\tilde{\rho}}] = \mathbf{G}_1 \oplus (n_V + 1)\mathbf{V}_1 \oplus (2n_H + n_V)\mathbf{C}_1 \tag{82}$$

with symmetries

$$[\mathfrak{so}(2)_l \oplus \mathfrak{u}(1)_R] \oplus [\mathfrak{so}(2)_r \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(n_V) \oplus \mathfrak{u}(n_H - 1)] \to \mathfrak{so}(2)_{st} \oplus [\mathfrak{u}(1)_R \oplus \mathfrak{u}(2) \oplus \mathfrak{u}(n_V) \oplus \mathfrak{u}(n_H)] \tag{83}$$

in agreement with (75). Note, in (80) and (83) n_V and n_H are interchanged precisely because $\mathbf{V}_2 \otimes \tilde{A} = \mathbf{G}_2 \oplus \mathbf{V}_2$ generates an extra vector multiplet whereas $\mathbf{V}_1 \otimes \tilde{A} = \mathbf{G}_1 \oplus \mathbf{C}_1$ generates an extra chiral multiplet.

IV. CONCLUSIONS

We have shown that all non-maximal supergravities theories in $3 \leq D \leq 6$ that are the product of two super Yang-Mills multiplets have a twin supergravity. Moreover, it has been shown that the parent supergravity and its twins are all related through their Yang-Mills factorisations in a uniform manner. As far as we are aware the matter coupled $\mathcal{N}_-$ twins generated this way have not been double-copy constructed previously, so that we add to the already substantial list of supergravities admitting a Yang-Mills factorisation.

In the course of studying the twin pyramid it has become clear that the factorisation of matter coupled supergravity theories is not necessarily unique. Note, it had already been previously observed that supergravities can admit factorisations into alternative multiplets, for example $D = 3, \mathcal{N} = 16$ supergravity is the square of both $\mathcal{N} = 8$ Yang-Mills theory and Bagger-Lambert-Gustavsson Chern-Simons theory [14]. In future work [78] we will give a complete classification, using symmetry generating constraints, of all alternative factorisations and double-copy constructible theories under the assumption that the (super)gravity scalar manifold is symmetric.

Considering the off-shell symmetries of the double-copy including fundamental matter multiplets led us to introduce a bi-fundamental scalar field that couples to a bi-adjoint scalar field through a cubic interaction. Interestingly, it seems that bi-adjoint/fundamental scalar theory yields the zeroth-copy of Yang-Mills-matter amplitudes at tree-level, suggesting a generalisation of the spin 2,1,0 CHY formulae of [34] to include non-adjoint fields.

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Appendix A: Irreducible Riemannian globally symmetric Kähler and Quaternionic Manifolds

Kahler manifolds:

$$\begin{aligned}
&\frac{\mathrm{SU}(p,q)}{\mathrm{SU}(p)\times\mathrm{SU}(q)\times\mathrm{U}(1)}, && \frac{\mathrm{SO}(p,2)}{\mathrm{SO}(p)\times\mathrm{U}(1)}, && \frac{\mathrm{Sp}(p,\mathbb{R})}{\mathrm{SU}(p)\times\mathrm{U}(1)} \\
&\frac{\mathrm{SO}^*(2p)}{\mathrm{SU}(p)\times\mathrm{U}(1)}, && \frac{E_{6(-14)}}{\mathrm{SO}(10)\times\mathrm{U}(1)}, && \frac{E_{7(-25)}}{E_6\times\mathrm{U}(1)}
\end{aligned} \tag{A1}$$

Quaternionic manifolds:

$$
\frac{\mathrm{SU}(p,2)}{\mathrm{SU}(p)\times\mathrm{SU}(2)\times\mathrm{U}(1)}\,,\qquad \frac{\mathrm{SO}(p,4)}{\mathrm{SO}(p)\times\mathrm{SO}(4)}\,,\qquad \frac{\mathrm{USp}(p,1)}{\mathrm{USp}(p)\times\mathrm{U}(1)}
$$
$$
\frac{G_2}{\mathrm{SU}(2)\times\mathrm{SU}(2)}\,,\qquad \frac{F_{4(4)}}{\mathrm{USp}(3)\times\mathrm{U}(1)}\,,\qquad \frac{E_{6(2)}}{\mathrm{SU}(6)\times\mathrm{SU}(2)}
$$
$$
\frac{E_{7(-5)}}{\mathrm{SO}(12)\times\mathrm{SU}(2)}\,,\qquad \frac{E_{8(-24)}}{E_7\times\mathrm{SU}(2)} \tag{A2}
$$

Appendix B: Bi-adjoint coupled to bi-fundamental scalar theory: a 5-point example

In section III we discussed how, starting with amplitudes for adjoint and fundamental fields, one can take the so-called "zeroth-copy" and generate amplitudes for a bi-adjoint scalar theory coupled to a bi-fundamental scalar. Here we give a 5-point example with two quark flavours, the double-copy of which was treated in [15]. This nicely illustrates the point that the bi-adjoint/fundamental scalar theory can be embellished to capture flavour groups and other structural features appearing in the gauge theory. We start with the scattering amplitude of two quark-antiquark pairs, with distinct flavours, with a single gluon. The two flavours reduces the Feynman diagrams to the five presented in Figure 3.

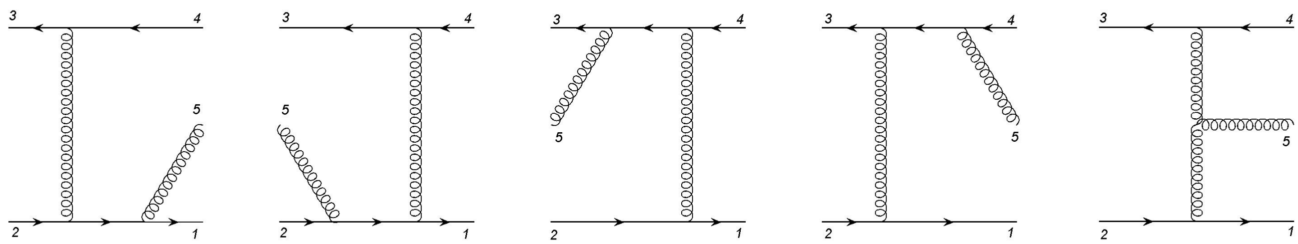

FIG. 3. Tree-level Feynman diagrams for the 5-point quark-antiquark-quark'-antiquark'-gluon interaction.

We use the labelling $(1, \bar{u}_1(k_1),^i)$ and $(2, v_2(k_2),_j)$ for one of the quark-antiquark pairs, then $(3, \bar{u}_3(k_3),^k)$ and $(4, v_4(k_4),_l)$ for the other (possessing a different flavour) and finally $(5, \varepsilon_5^\mu(k_5), a)$ for the gluon. The colour and kinematic factors for these diagrams are then

$$
\begin{aligned}
c_1\times n_1 &= -i\ [T^a]_i{}^m[T^b]_m^{\,j}[T^b]_k^{\,l} && \times\quad \bar{u}_3\gamma^\nu v_4\ \bar{u}_1\varepsilon_5^\mu\gamma_\mu\gamma^\rho(k_1+k_5)_\rho\gamma_\nu v_2\\
c_2\times n_2 &= -i\ [T^b]_i{}^m[T^a]_m^{\,j}[T^b]_k^{\,l} && \times\quad \bar{u}_3\gamma^\nu v_4\ \bar{u}_1\gamma_\nu\gamma^\rho(-k_2-k_5)_\rho\varepsilon_5^\mu\gamma_\mu v_2\\
c_3\times n_3 &= -i\ [T^a]_k{}^m[T^b]_m^{\,l}[T^b]_i^{\,j} && \times\quad \bar{u}_3\gamma_\mu\varepsilon_5^\mu\gamma^\rho(k_3+k_5)_\rho\gamma^\nu v_4\ \bar{u}_1\gamma_\nu v_2\\
c_4\times n_4 &= -i\ [T^b]_k{}^m[T^a]_m^{\,l}[T^b]_i^{\,j} && \times\quad \bar{u}_3\gamma^\nu\gamma^\rho(-k_4-k_5)_\rho\varepsilon_5^\mu\gamma_\mu v_4\ \bar{u}_1\gamma_\nu v_2\\
c_5\times n_5 &= \qquad f^{abc}[T^b]_i{}^j[T^c]_k{}^l && \times\quad \bar{u}_3\gamma^\nu v_4\ [\eta_{\mu\rho}(k_5-q)_\nu+\eta_{\rho\nu}(q-p)_\mu+\eta_{\nu\mu}(p-k_5)_\rho]\varepsilon_5^\mu\ \bar{u}_1\gamma^\rho v_2
\end{aligned} \tag{B1}
$$

where, for the last diagram, $q = k_1 + k_2$ and $p = k_3 + k_4$. We know that these numerators automatically satisfy BCJ relations, so we can proceed as in the 4-point example and replace the kinematic factors with a second set of colour factors,

$$
\begin{aligned}
c_1\times \tilde{c}_1 &= -i\ [T^a]_i{}^m[T^b]_m^{\,j}[T^b]_k^{\,l} && \times\quad [\tilde{T}^{\tilde{a}}]_{\tilde{i}}{}^{\tilde{m}}[\tilde{T}^{\tilde{b}}]_{\tilde{m}}^{\,\tilde{j}}[\tilde{T}^{\tilde{b}}]_{\tilde{k}}^{\,\tilde{l}}\\
c_2\times \tilde{c}_2 &= -i\ [T^b]_i{}^m[T^a]_m^{\,j}[T^b]_k^{\,l} && \times\quad [\tilde{T}^{\tilde{b}}]_{\tilde{i}}{}^{\tilde{m}}[\tilde{T}^{\tilde{a}}]_{\tilde{m}}^{\,\tilde{j}}[\tilde{T}^{\tilde{b}}]_{\tilde{k}}^{\,\tilde{l}}\\
c_3\times \tilde{c}_3 &= -i\ [T^a]_k{}^m[T^b]_m^{\,l}[T^b]_i^{\,j} && \times\quad [\tilde{T}^{\tilde{a}}]_{\tilde{k}}{}^{\tilde{m}}[\tilde{T}^{\tilde{b}}]_{\tilde{m}}^{\,\tilde{l}}[\tilde{T}^{\tilde{b}}]_{\tilde{i}}^{\,\tilde{j}}\\
c_4\times \tilde{c}_4 &= -i\ [T^b]_k{}^m[T^a]_m^{\,l}[T^b]_i^{\,j} && \times\quad [\tilde{T}^{\tilde{b}}]_{\tilde{k}}{}^{\tilde{m}}[\tilde{T}^{\tilde{a}}]_{\tilde{m}}^{\,\tilde{l}}[\tilde{T}^{\tilde{b}}]_{\tilde{i}}^{\,\tilde{j}}\\
c_5\times \tilde{c}_5 &= f^{abc}[T^b]_i{}^j[T^c]_k{}^l && \times\quad \tilde{f}^{\tilde{a}\tilde{b}\tilde{c}}[\tilde{T}^{\tilde{b}}]_{\tilde{i}}{}^{\tilde{j}}[\tilde{T}^{\tilde{c}}]_{\tilde{k}}{}^{\tilde{l}}
\end{aligned} \tag{B2}
$$

and we see that these reproduce the amplitude for a bi-adjoint scalar theory coupled to a pair of bi-fundamental scalars, with a non-trivial flavour group (see Figure 4). The cubic interactions for this theory are described by the

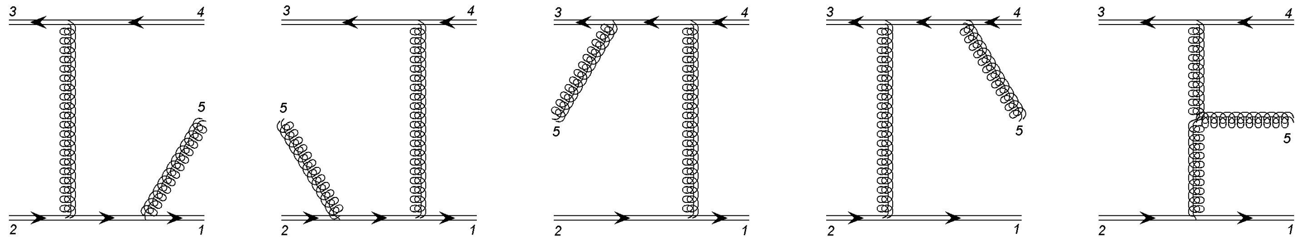

FIG. 4. Double-line tree-level Feynman diagrams for the 5-point (bi-fund.)-(bi-fund.)-(bi-fund'.)-(bi-fund'.)-(bi-adj.) interaction for the scalar theory in (B3). The curly (straight) double-lines represent the bi-adjoint (bi-fundamental) representation of the global $G \times \tilde{G}$ symmetry. Each diagram is the double-copy of the corresponding gluon-quark diagram shawn of its kinematic data.

Lagrangian

$$\mathcal{L}_{\text{bi-adj-fund}} = -\frac{1}{2}\partial_\mu \Phi_{a\tilde{b}} \partial^\mu \Phi^{a\tilde{b}} - \frac{1}{2}\partial_\mu \Phi^{\alpha}_{\tilde{i}i} \partial^\mu \Phi^{i\tilde{i}}_{\alpha} + \frac{g}{6}\left(f_{abc}\tilde{f}_{\tilde{a}\tilde{b}\tilde{c}} \Phi^{a\tilde{a}}\Phi^{b\tilde{b}}\Phi^{c\tilde{c}} + i[T^a]_i{}^j [\tilde{T}^{\tilde{a}}]_{\tilde{i}}{}^{\tilde{j}} \Phi_{a\tilde{a}} \Phi^{i\tilde{i}}_{\alpha} \Phi^{\alpha}_{j\tilde{j}} \right), \tag{B3}$$

where α denotes the representation of the flavour group, which the scalars have inherited from the original theory. We note that the replacement rules postulated in (28) and (29) still hold since the Feynman diagrams are already BCJ-duality respecting (i.e. there are no four-point contact terms in this example).

[1] Yacine Dolivet, Bernard Julia, and Costas Kounnas, "Magic N=2 supergravities from hyper-free superstrings," JHEP **02**, 097 (2008), arXiv:0712.2867 [hep-th].
[2] Massimo Bianchi and Sergio Ferrara, "Enriques and Octonionic Magic Supergravity Models," JHEP **02**, 054 (2008), arXiv:0712.2976 [hep-th].
[3] Diederik Roest and Henning Samtleben, "Twin Supergravities," Class. Quant. Grav. **26**, 155001 (2009), arXiv:0904.1344 [hep-th].
[4] M. J. Duff and S. Ferrara, "Generalized mirror symmetry and trace anomalies," Class.Quant.Grav. **28**, 065005 (2011), arXiv:1009.4439 [hep-th].
[5] A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes, and S. Nagy, "Yang-Mills origin of gravitational symmetries," Phys.Rev.Lett. **113**, 231606 (2014), arXiv:1408.4434 [hep-th].
[6] L. Borsten and M. J. Duff, "Gravity as the square of Yang–Mills?" Phys. Scripta **90**, 108012 (2015), arXiv:1602.08267 [hep-th].
[7] H. Kawai, D.C. Lewellen, and S.H.H. Tye, "A Relation Between Tree Amplitudes of Closed and Open Strings," Nucl.Phys. **B269**, 1 (1986).
[8] Z. Bern, J.J.M. Carrasco, and Henrik Johansson, "New Relations for Gauge-Theory Amplitudes," Phys.Rev. **D78**, 085011 (2008), arXiv:0805.3993 [hep-ph].
[9] Zvi Bern, Tristan Dennen, Yu-tin Huang, and Michael Kiermaier, "Gravity as the Square of Gauge Theory," Phys.Rev. **D82**, 065003 (2010), arXiv:1004.0693 [hep-th].
[10] Zvi Bern, John Joseph M. Carrasco, and Henrik Johansson, "Perturbative Quantum Gravity as a Double Copy of Gauge Theory," Phys.Rev.Lett. **105**, 061602 (2010), arXiv:1004.0476 [hep-th].
[11] John Joseph M. Carrasco, Marco Chiodaroli, Murat Günaydin, and Radu Roiban, "One-loop four-point amplitudes in pure and matter-coupled N = 4 supergravity," JHEP **1303**, 056 (2013), arXiv:1212.1146 [hep-th].
[12] Poul H. Damgaard, Rijun Huang, Thomas Sondergaard, and Yang Zhang, "The Complete KLT-Map Between Gravity and Gauge Theories," JHEP **1208**, 101 (2012), arXiv:1206.1577 [hep-th].
[13] Yu-tin Huang and Henrik Johansson, "Equivalent D=3 Supergravity Amplitudes from Double Copies of Three-Algebra and Two-Algebra Gauge Theories," Phys.Rev.Lett. **110**, 171601 (2013), arXiv:1210.2255 [hep-th].
[14] Till Bargheer, Song He, and Tristan McLoughlin, "New Relations for Three-Dimensional Supersymmetric Scattering Amplitudes," Phys.Rev.Lett. **108**, 231601 (2012), arXiv:1203.0562 [hep-th].
[15] Henrik Johansson and Alexander Ochirov, "Pure Gravities via Color-Kinematics Duality for Fundamental Matter," JHEP **11**, 046 (2015), arXiv:1407.4772 [hep-th].
[16] Marco Chiodaroli, Murat Günaydin, Henrik Johansson, and Radu Roiban, "Scattering amplitudes in $\mathcal{N} = 2$ Maxwell-Einstein and Yang-Mills/Einstein supergravity," JHEP **01**, 081 (2015), arXiv:1408.0764 [hep-th].
[17] Marco Chiodaroli, Murat Gunaydin, Henrik Johansson, and Radu Roiban, "Spontaneously Broken Yang-Mills-Einstein Supergravities as Double Copies," (2015), arXiv:1511.01740 [hep-th].
[18] Marco Chiodaroli, Murat Gunaydin, Henrik Johansson, and Radu Roiban, "Complete construction of magical, symmetric and homogeneous N=2 supergravities as double copies of gauge theories," (2015), arXiv:1512.09130 [hep-th].

[19] Marco Chiodaroli, “Simplifying amplitudes in Maxwell-Einstein and Yang-Mills-Einstein supergravities,” (2016) arXiv:1607.04129 [hep-th].
[20] Henriette Elvang and Yu-tin Huang, “Scattering Amplitudes,” (2013), arXiv:1308.1697 [hep-th].
[21] John Joseph M. Carrasco, “Gauge and Gravity Amplitude Relations,” (2015), arXiv:1506.00974 [hep-th].
[22] Panagiotis Katsaroumpas, Bill Spence, and Gabriele Travaglini, “One-loop N=8 supergravity coefficients from N=4 super Yang-Mills,” JHEP **0908**, 096 (2009), arXiv:0906.0521 [hep-th].
[23] Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson, and R. Roiban, “The Ultraviolet Behavior of N=8 Supergravity at Four Loops,” Phys. Rev. Lett. **103**, 081301 (2009), arXiv:0905.2326 [hep-th].
[24] Zvi Bern, Scott Davies, Tristan Dennen, Alexander V. Smirnov, and Vladimir A. Smirnov, “Ultraviolet Properties of N=4 Supergravity at Four Loops,” Phys. Rev. Lett. **111**, 231302 (2013), arXiv:1309.2498 [hep-th].
[25] J. J. M. Carrasco, R. Kallosh, R. Roiban, and A. A. Tseytlin, “On the U(1) duality anomaly and the S-matrix of N=4 supergravity,” JHEP **07**, 029 (2013), arXiv:1303.6219 [hep-th].
[26] Zvi Bern, Scott Davies, Tristan Dennen, Yu-tin Huang, and Josh Nohle, “Color-Kinematics Duality for Pure Yang-Mills and Gravity at One and Two Loops,” Phys. Rev. **D92**, 045041 (2015), arXiv:1303.6605 [hep-th].
[27] Zvi Bern, Scott Davies, and Tristan Dennen, “The Ultraviolet Structure of Half-Maximal Supergravity with Matter Multiplets at Two and Three Loops,” Phys. Rev. **D88**, 065007 (2013), arXiv:1305.4876 [hep-th].
[28] Zvi Bern, Scott Davies, and Tristan Dennen, “The Ultraviolet Critical Dimension of Half-Maximal Supergravity at Three Loops,” (2014), arXiv:1412.2441 [hep-th].
[29] Zvi Bern, Scott Davies, and Tristan Dennen, “Enhanced ultraviolet cancellations in $\mathcal{N} = 5$ supergravity at four loops,” Phys.Rev. **D90**, 105011 (2014), arXiv:1409.3089 [hep-th].
[30] Zvi Bern, Clifford Cheung, Huan-Hang Chi, Scott Davies, Lance Dixon, and Josh Nohle, “Evanescent Effects Can Alter Ultraviolet Divergences in Quantum Gravity without Physical Consequences,” Phys. Rev. Lett. **115**, 211301 (2015), arXiv:1507.06118 [hep-th].
[31] Zvi Bern, Enrico Herrmann, Sean Litsey, James Stankowicz, and Jaroslav Trnka, “Logarithmic Singularities and Maximally Supersymmetric Amplitudes,” JHEP **06**, 202 (2015), arXiv:1412.8584 [hep-th].
[32] N.E.J. Bjerrum-Bohr and Oluf Tang Engelund, “Gravitino Interactions from Yang-Mills Theory,” Phys.Rev. **D81**, 105009 (2010), arXiv:1002.2279 [hep-th].
[33] Zvi Bern, Scott Davies, and Josh Nohle, “Double-Copy Constructions and Unitarity Cuts,” Phys. Rev. **D93**, 105015 (2016), arXiv:1510.03448 [hep-th].
[34] Freddy Cachazo, Song He, and Ellis Ye Yuan, “Scattering of Massless Particles: Scalars, Gluons and Gravitons,” JHEP **1407**, 033 (2014), arXiv:1309.0885 [hep-th].
[35] Louise Dolan and Peter Goddard, “Proof of the Formula of Cachazo, He and Yuan for Yang-Mills Tree Amplitudes in Arbitrary Dimension,” JHEP **1405**, 010 (2014), arXiv:1311.5200 [hep-th].
[36] Freddy Cachazo, Song He, and Ellis Ye Yuan, “Scattering Equations and Matrices: From Einstein To Yang-Mills, DBI and NLSM,” JHEP **07**, 149 (2015), arXiv:1412.3479 [hep-th].
[37] Stephen G. Naculich, “Scattering equations and BCJ relations for gauge and gravitational amplitudes with massive scalar particles,” JHEP **09**, 029 (2014), arXiv:1407.7836 [hep-th].
[38] Stephen G. Naculich, “CHY representations for gauge theory and gravity amplitudes with up to three massive particles,” JHEP **05**, 050 (2015), arXiv:1501.03500 [hep-th].
[39] Yi-Jian Du and Chih-Hao Fu, “Explicit BCJ numerators of nonlinear sigma model,” (2016), arXiv:1606.05846 [hep-th].
[40] Yi-jian Du, Fei Teng, and Yong-shi Wu, “CHY formula and MHV amplitudes,” JHEP **05**, 086 (2016), arXiv:1603.08158 [hep-th].
[41] Dhritiman Nandan, Jan Plefka, Oliver Schlotterer, and Congkao Wen, “Einstein-Yang-Mills from pure Yang-Mills amplitudes,” (2016), arXiv:1607.05701 [hep-th].
[42] N. E. J. Bjerrum-Bohr, Jacob L. Bourjaily, Poul H. Damgaard, and Bo Feng, “Manifesting Color-Kinematics Duality in the Scattering Equation Formalism,” (2016), arXiv:1608.00006 [hep-th].
[43] Ricardo Monteiro and Donal O’Connell, “The Kinematic Algebra From the Self-Dual Sector,” JHEP **1107**, 007 (2011), arXiv:1105.2565 [hep-th].
[44] N.E.J. Bjerrum-Bohr, Poul H. Damgaard, Ricardo Monteiro, and Donal O’Connell, “Algebras for Amplitudes,” JHEP **1206**, 061 (2012), arXiv:1203.0944 [hep-th].
[45] Ricardo Monteiro and Donal O’Connell, “The Kinematic Algebras from the Scattering Equations,” JHEP **1403**, 110 (2014), arXiv:1311.1151 [hep-th].
[46] Rutger H. Boels, Reinke Sven Isermann, Ricardo Monteiro, and Donal O’Connell, “Colour-Kinematics Duality for One-Loop Rational Amplitudes,” JHEP **1304**, 107 (2013), arXiv:1301.4165 [hep-th].
[47] Chih-Hao Fu and Kirill Krasnov, “Colour-Kinematics duality and the Drinfeld double of the Lie algebra of diffeomorphisms,” (2016), arXiv:1603.02033 [hep-th].
[48] N. E. J. Bjerrum-Bohr, Poul H. Damgaard, and Pierre Vanhove, “Minimal Basis for Gauge Theory Amplitudes,” Phys. Rev. Lett. **103**, 161602 (2009), arXiv:0907.1425 [hep-th].
[49] S. Stieberger, “Open & Closed vs. Pure Open String Disk Amplitudes,” (2009), arXiv:0907.2211 [hep-th].
[50] Carlos R. Mafra, Oliver Schlotterer, and Stephan Stieberger, “Explicit BCJ Numerators from Pure Spinors,” JHEP **07**, 092 (2011), arXiv:1104.5224 [hep-th].
[51] Johannes Broedel and Lance J. Dixon, “Color-kinematics duality and double-copy construction for amplitudes from higher-dimension operators,” JHEP **10**, 091 (2012), arXiv:1208.0876 [hep-th].
[52] Johannes Broedel, Oliver Schlotterer, and Stephan Stieberger, “Polylogarithms, Multiple Zeta Values and Superstring

Amplitudes," Fortsch. Phys. **61**, 812–870 (2013), arXiv:1304.7267 [hep-th].
[53] Carlos R. Mafra and Oliver Schlotterer, "Towards one-loop SYM amplitudes from the pure spinor BRST cohomology," Fortsch. Phys. **63**, 105–131 (2015), arXiv:1410.0668 [hep-th].
[54] Carlos R. Mafra and Oliver Schlotterer, "Berends-Giele recursions and the BCJ duality in superspace and components," (2015), arXiv:1510.08846 [hep-th].
[55] Carlos R. Mafra and Oliver Schlotterer, "Two-loop five-point amplitudes of super Yang-Mills and supergravity in pure spinor superspace," JHEP **10**, 124 (2015), arXiv:1505.02746 [hep-th].
[56] Song He, Ricardo Monteiro, and Oliver Schlotterer, "String-inspired BCJ numerators for one-loop MHV amplitudes," JHEP **01**, 171 (2016), arXiv:1507.06288 [hep-th].
[57] Oliver Schlotterer, "Amplitude relations in heterotic string theory and Einstein-Yang-Mills," (2016), arXiv:1608.00130 [hep-th].
[58] John Joseph M. Carrasco, Carlos R. Mafra, and Oliver Schlotterer, "Abelian Z-theory: NLSM amplitudes and alpha'-corrections from the open string," (2016), arXiv:1608.02569 [hep-th].
[59] Lionel Mason and David Skinner, "Ambitwistor strings and the scattering equations," JHEP **1407**, 048 (2014), arXiv:1311.2564 [hep-th].
[60] Yvonne Geyer, Arthur E. Lipstein, and Lionel J. Mason, "Ambitwistor Strings in Four Dimensions," Phys. Rev. Lett. **113**, 081602 (2014), arXiv:1404.6219 [hep-th].
[61] Eduardo Casali, Yvonne Geyer, Lionel Mason, Ricardo Monteiro, and Kai A. Roehrig, "New Ambitwistor String Theories," JHEP **11**, 038 (2015), arXiv:1506.08771 [hep-th].
[62] Yvonne Geyer, Lionel Mason, Ricardo Monteiro, and Piotr Tourkine, "Loop Integrands for Scattering Amplitudes from the Riemann Sphere," Phys. Rev. Lett. **115**, 121603 (2015), arXiv:1507.00321 [hep-th].
[63] Yvonne Geyer, Lionel Mason, Ricardo Monteiro, and Piotr Tourkine, "One-loop amplitudes on the Riemann sphere," JHEP **03**, 114 (2016), arXiv:1511.06315 [hep-th].
[64] Yvonne Geyer, Lionel Mason, Ricardo Monteiro, and Piotr Tourkine, "Two-Loop Scattering Amplitudes from the Riemann Sphere," (2016), arXiv:1607.08887 [hep-th].
[65] Marco Chiodaroli, Murat Gunaydin, and Radu Roiban, "Superconformal symmetry and maximal supergravity in various dimensions," JHEP **1203**, 093 (2012), arXiv:1108.3085 [hep-th].
[66] L. Borsten, M. J. Duff, L. J. Hughes, and S. Nagy, "A magic square from Yang-Mills squared," Phys.Rev.Lett. **112**, 131601 (2014), arXiv:1301.4176 [hep-th].
[67] A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes, and S. Nagy, "Super Yang-Mills, division algebras and triality," JHEP **1408**, 080 (2014), arXiv:1309.0546 [hep-th].
[68] A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes, and S. Nagy, "A magic pyramid of supergravities," JHEP **1404**, 178 (2014), arXiv:1312.6523 [hep-th].
[69] S. Nagy, "Chiral Squaring," JHEP **07**, 142 (2016), arXiv:1412.4750 [hep-th].
[70] A. Anastasiou, L. Borsten, M. J. Hughes, and S. Nagy, "Global symmetries of Yang-Mills squared in various dimensions," JHEP **148**, 1601 (2016), arXiv:1502.05359 [hep-th].
[71] Rutger H. Boels and Ricardo Medina, "Graviton and gluon scattering from first principles," (2016), arXiv:1607.08246 [hep-th].
[72] Ricardo Monteiro, Donal O'Connell, and Chris D. White, "Black holes and the double copy," JHEP **1412**, 056 (2014), arXiv:1410.0239 [hep-th].
[73] Andrés Luna, Ricardo Monteiro, Donal O'Connell, and Chris D. White, "The classical double copy for Taub–NUT spacetime," Phys. Lett. **B750**, 272–277 (2015), arXiv:1507.01869 [hep-th].
[74] Andréas Luna, Ricardo Monteiro, Isobel Nicholson, Donal O'Connell, and Chris D. White, "The double copy: Bremsstrahlung and accelerating black holes," JHEP **06**, 023 (2016), arXiv:1603.05737 [hep-th].
[75] Chris D. White, "Exact solutions for the biadjoint scalar field," (2016), arXiv:1606.04724 [hep-th].
[76] G. L. Cardoso, S. Nagy, and Suresh Nampuri, "A double copy for $\mathcal{N} = 2$ supergravity: a linearised tale told on-shell," (2016), arXiv:1609.05022 [hep-th].
[77] Laura Andrianopoli, Riccardo D'Auria, Sergio Ferrara, and Mario Trigiante, "Black-hole attractors in N=1 supergravity," JHEP **07**, 019 (2007), arXiv:hep-th/0703178 [HEP-TH].
[78] A. Anastasiou, L. Borsten, M. J. Duff, M. J. Hughes, A. Marrani, S. Nagy, and M. Zoccali, "A plethora of alternative yang-mills factorisations," (2016), to appear.
[79] Andrew Hodges, "New expressions for gravitational scattering amplitudes," Journal of High Energy Physics **1307** (2013), 10.1007/JHEP07(2013)075, arXiv:1108.2227 [hep-th].
[80] E. Cremmer, Joel Scherk, and S. Ferrara, "SU(4) Invariant Supergravity Theory," Phys. Lett. **B74**, 61–64 (1978).
[81] Robert B. Brown, "Groups of type E_7," J. Reine Angew. Math. **236**, 79–102 (1969).
[82] Sergio Ferrara, Alessandra Gnecchi, and Alessio Marrani, "d=4 Attractors, Effective Horizon Radius and Fake Supergravity," Phys. Rev. **D78**, 065003 (2008), arXiv:0806.3196 [hep-th].
[83] J.F. Luciani, "Coupling of O(2) Supergravity with Several Vector Multiplets," Nucl.Phys. **B132**, 325 (1978).
[84] Sergio Ferrara, Alessio Marrani, and Emanuele Orazi, "Split Attractor Flow in N=2 Minimally Coupled Supergravity," Nucl.Phys. **B846**, 512–541 (2011), arXiv:1010.2280 [hep-th].
[85] Sergio Ferrara, Renata Kallosh, and Alessio Marrani, "Degeneration of Groups of Type E7 and Minimal Coupling in Supergravity," JHEP **1206**, 074 (2012), arXiv:1202.1290 [hep-th].
[86] S. Cecotti, S. Ferrara, and L. Girardello, "Geometry of Type II Superstrings and the Moduli of Superconformal Field Theories," Int. J. Mod. Phys. **A4**, 2475 (1989).

[87] B. de Wit, F. Vanderseypen, and Antoine Van Proeyen, "Symmetry structure of special geometries," Nucl. Phys. **B400**, 463–524 (1993), arXiv:hep-th/9210068 [hep-th].
[88] Anna Ceresole, Sergio Ferrara, and Alessio Marrani, "4d/5d Correspondence for the Black Hole Potential and its Critical Points," Class. Quant. Grav. **24**, 5651–5666 (2007), arXiv:0707.0964 [hep-th].
[89] L. Castellani, Anna Ceresole, S. Ferrara, R. D'Auria, P. Fre, *et al.*, "The Complete N=3 Matter Coupled Supergravity," Nucl.Phys. **B268**, 317 (1986).
[90] Murat Günaydin, G. Sierra, and P. K. Townsend, "The geometry of $N = 2$ Maxwell-Einstein supergravity and Jordan algebras," Nucl. Phys. **B242**, 244 (1984).
[91] B. de Wit and Antoine Van Proeyen, "Potentials and Symmetries of General Gauged N=2 Supergravity: Yang-Mills Models," Nucl. Phys. **B245**, 89–117 (1984).
[92] Laura Andrianopoli, Riccardo D'Auria, and Sergio Ferrara, "U duality and central charges in various dimensions revisited," Int. J. Mod. Phys. **A13**, 431–490 (1998), arXiv:hep-th/9612105 [hep-th].
[93] Marco Chiodaroli, Qingjun Jin, and Radu Roiban, "Color/kinematics duality for general abelian orbifolds of N=4 super Yang-Mills theory," JHEP **01**, 152 (2014), arXiv:1311.3600 [hep-th].
[94] Murat Günaydin, G. Sierra, and P. K. Townsend, "Exceptional supergravity theories and the magic square," Phys. Lett. **B133**, 72 (1983).
[95] Murat Günaydin, G. Sierra, and P. K. Townsend, "Gauging the $d = 5$ Maxwell-Einstein supergravity theories: More on Jordan algebras," Nucl. Phys. **B253**, 573 (1985).
[96] Yu-tin Huang and Arthur E. Lipstein, "Amplitudes of 3D and 6D Maximal Superconformal Theories in Supertwistor Space," JHEP **1010**, 007 (2010), arXiv:1004.4735 [hep-th].
[97] Bartlomiej Czech, Yu-tin Huang, and Moshe Rozali, "Chiral three-point interactions in 5 and 6 dimensions," JHEP **1210**, 143 (2012), arXiv:1110.2791 [hep-th].
[98] C.M. Hull, "Strongly coupled gravity and duality," Nucl.Phys. **B583**, 237–259 (2000), arXiv:hep-th/0004195 [hep-th].
[99] C.M. Hull, "Symmetries and compactifications of (4,0) conformal gravity," JHEP **0012**, 007 (2000), arXiv:hep-th/0011215 [hep-th].